# RESEARCH STATEMENT 

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My research is in Geometric Group Theory. I have broad interests but I am especially interested in constructing and computing invariants which detect coarse manifestations of "non-positive curvature" with a particular emphasis on stable commutator length and bounded cohomology.

## 1. Stable Commutator Length

Stable commutator length (scl) is an invariant of elements in the commutator subgroup of a group $G$; it has intriguing algebraic, geometric and analytic features. If $\gamma$ is a loop in a topological space $X$ such that $[\gamma] \in \pi_{1}(X)$ lies in the commutator subgroup, then the commutator length of $\gamma$ is the smallest genus of a surface that bounds $\gamma$ and similarly its stable commutator length is the infimum $\operatorname{scl}([\gamma])=\inf _{S} \frac{\chi(S)}{2 n(S)}$ over "admissible" surfaces $S$ which map to $X$ such that the boundary of $S$ maps to $\gamma$ with degree $n(S)$. Algebraically, commutator (resp. stable commutator) length counts the number of factors needed to express an element $g \in[G, G]$ (resp. its powers) as a product of commutators.

There is an alternative way to compute stable commutator length via quasimorphisms. A quasimorphism is a map $\phi: G \rightarrow \mathbb{R}$ such that there is a constant $C>0$ for which $|\phi(g)+\phi(h)-\phi(g h)| \leq C$ holds for all $g, h \in G$. The smallest such $C$ is called the defect of $\phi$ and denoted by $D(\phi)$. A quasimorphism is homogeneous, if it restricts to a homomorphism on cyclic subgroups. Bavards Duality Theorem [Bav91] asserts that $\operatorname{scl}(g)=\sup _{\phi} \frac{\phi(g)}{2 D(\phi)}$ where the supremum is taken over all homogeneous quasimorphisms $\phi$. For any element $g \in[G, G]$ there is an extremal homogeneous quasimorphism $\phi$ which realises this supremum. However, explicit extremal quasimorphisms have been found only in very few cases. In [Heua] I gave the first explicit construction of infinitely many extremal quasimorphisms $\phi_{w}$ on non-abelian free groups. These are derived from Brooks quasimorphisms by an iterative algebraic reduction algorithm.
Theorem A. [Heua] Let $F$ be a non-abelian free group. For all $w \in[F, F]$ with $\operatorname{scl}(w)=$ $1 / 2$ the explicit quasimorphism $\phi_{w}: F \rightarrow \mathbb{R}$ is extremal.

These quasimorphisms $\phi_{w}$ are a crucial tool in the proof of Theorem C below.
1.1. Gaps in Stable Commutator length. An algebraic manifestation of the thickthin decomposition of hyperbolic manifolds with fundamental group $G$ is a gap in stable commutator length. There is a constant $C>0$ such that for all $g \in[G, G]$ either $\operatorname{scl}(g) \geq$ $C$ holds, or $\operatorname{scl}(g)=0$ for "trivial" reasons. Such a gap exists more generally for many classes of groups associated to non-positive curvature, such as hyperbolic groups (see [CF10]), mapping class groups (see [BBF16]), Baumslag-Solitar groups (see [CFL16]),


Figure 1. Histogram of scl for 30,000 random words $g \in \mathbb{F}_{2}^{\prime}$ of length 24. The $x$-axis records values of stable commutator length the $y$-axis the occurences. The spike at 1 has been truncated. Calculations were done using scallop; see [Cal].
certain amalgamated free products [CF10], [CFL16] and certain free products [Che18]. For free non-abelian groups $F$ this gap is exactly 1/2; see [DH91], and Figure 1. By the monotonicity of scl, every element $g \in[G, G]$ in a group which is mapped to a non-trivial element in a free group $F$ also satisfies $\operatorname{scl}(g) \geq 1 / 2$. By generalising homomorphisms $G \rightarrow F$ I found a new criterion for gaps of $1 / 2$. By applying this criterion I prove:
Theorem B. [Heua] Let $A, B, C$ be groups, $\kappa_{A}: C \hookrightarrow A$ and $\kappa_{B}: C \hookrightarrow B$ injections and suppose both $\kappa_{A}(C)<A$ and $\kappa_{B}(C)<B$ are left-relatively convex. If $g \in A \star_{C} B$ does not conjugate into one of the factors then there is an explicit homogeneous quasimorphism $\phi: A \star_{C} B \rightarrow \mathbb{R}$ such that $\phi(g) \geq 1$ and $D(\phi)=1$. If $g$ is in the commutator subgroup then $\operatorname{scl}(g) \geq 1 / 2$.
Theorem C. [Heua] Every non-trivial element $g$ in the commutator subgroup of a subgroup of any right-angled Artin group $\mathrm{A}(\Gamma)$ satisfies $\operatorname{scl}(g) \geq 1 / 2$. This bound is sharp.

Subgroups of right-angled Artin groups are now known to be an extremely rich class, following the theory of special cube complexes. See [Wis09], [HW08], [Ago13] and [Bri13].

I could show that the boundary of all such constructed quasimorphisms is the orientation cocycle of an action $\rho: G \rightarrow \operatorname{Homeo}^{+}\left(S^{1}\right)$.

Question 1.1. Is there a more geometric way to construct these quasimorphisms? How is this construction related to the theory of continued fractions?
1.2. Surface maps. In joint work with Lvzhou Chen [CH], we could generalise Theorem B to graphs of groups obtain sharp gap results for a large class of groups, like graphs of groups and certain one-relator and 3 -manifold groups.
1.3. Gaps above $1 / 2$. Stable commutator length is rational and computable in free groups; [Cal09]. This allows one to conduct computer experiments on the distribution of stable commutator length and a mysterious distribution in the frequency of the values of stable commutator length emerges; see Figure 1.
Question 1.2. Explain the self-similarity features of Figure 1.

Question 1.3. Are there more gaps above $1 / 2$ ?
Experiments suggest that there is no element $g \in F$ such that $1 / 2<\operatorname{scl}(g)<7 / 12$. A first step in proving such a second gap would be to characterize those $g \in F^{\prime}$ with $\operatorname{scl}(g)=1 / 2$.
Conjecture 1.4. Let $g \in F^{\prime}$ be such that $\operatorname{scl}(g)=1 / 2$. Either $\operatorname{cl}(g)=1$ or there exists $t \in F$ with $\operatorname{cl}\left(g t g t^{-1}\right)=1$.

Giles Gardam and I verified the conjecture for all elemenets $g \in F^{\prime}$ of length less than 20.

## 2. Simplicial Volume of manifolds

The simplicial volume $\|M\|$ of an orientable closed connected manifold $M$ is a homotopy invariant that captures the complexity of representing fundamental classes by singular cycles with real coefficients. Simplicial volume is known to be positive in the presence of enough negative curvature [Gro82b, Thu97] and known to vanish in the presence of enough amenability [Gro82b, Iva85]. Moreover, it provides a topological lower bound for the minimal Riemannian volume (suitably normalised) in the case of smooth manifolds [Gro82b].

Until now, for large dimensions $d$, very little was known about the precise structure of the set $\mathrm{SV}(d) \subset \mathbb{R}_{\geq 0}$ of simplicial volumes of orientable closed connected $d$-manifolds. The set $\operatorname{SV}(d)$ is countable and closed under addition. However, the set of simplicial volumes is fully understood only in dimensions 2 and 3 with $\operatorname{SV}(2)=\mathbb{N}[4]$ and $\operatorname{SV}(3)=$ $\mathbb{N}\left[\left.\frac{\operatorname{vol}(M)}{v} \right\rvert\, M\right]$, where $M$ ranges over all complete finite-volume hyperbolic 3-manifolds with toroidal boundary and where $v>0$ is a constant.

This reveals that there is a gap of simplicial volume in dimensions 2 and 3: For $d \in\{2,3\}$ there is a constant $C_{d}>0$ such that the simplicial volume of an orientable closed connected $d$-manifold either vanishes or is at least $C_{d}$. It was an open question [Sam99, p. 550] whether such a gap exists in higher dimensions. For example, the previously lowest known simplicial volume of an orientable closed connected 4-manifold has been 24 [BK08].

In joint work with Clara Löh [HL] we showed that dimensions 2 and 3 are the only dimensions with such a gap.

Theorem D ([HL, Theorem A]). Let $d \geq 4$ be an integer. For every $\epsilon>0$ there is an orientable closed connected d-manifold $M$ such that $0<\|M\| \leq \epsilon$. Hence, the set of simplicial volumes of orientable closed connected d-manifolds is dense in $\mathbb{R}_{\geq 0}$.

In dimension 4, we get the following refinement of Theorem $D$.
Theorem $\mathbf{E}\left([\mathrm{HL}\right.$, Theorem B] $]$. For every $q \in \mathbb{Q}_{\geq 0}$ there is an orientable closed connected 4-manifold $M_{q}$ with $\left\|M_{q}\right\|=q$.

## 3. Constructions in Bounded Cohomology

A guiding theme in Geometric Group Theory is to classify groups by the geometry of the spaces they act on. In such classification schemes there are typically two extremes: amenable groups and various notions of "non-positive-" and "negative curvature"; see
[Bri06]. The bounded cohomology of a group $G$ picks out negative curvatured features of $G$. For example, $\mathrm{H}_{b}^{n}(G, \mathbb{R})$ is uncountable dimensional if $G$ is acylindrically hyperbolic and $n=2,3$ but $\mathrm{H}_{b}^{n}(G, \mathbb{R})=0$ for all $n \geq 1$ if $G$ is amenable; see [HO13], [FFPS17], [FPS15]. Bounded cohomology is intimately related to stable commutator length as the defect of quasimorphisms on $G$ are exactly the exact classes in $\mathrm{H}_{b}^{2}(G, \mathbb{R})$. The study of bounded cohomology in geometry and group theory was initiated by Gromov in [Gro82a] in connection to the minimal volume of manifolds. Since then bounded cohomology has emerged as an indepenedent research field with many applications.

Let $V$ be a normed $G$-module and let $\mathrm{H}^{n}(G, V)$ be the ordinary cohomology of $G$ with coefficients in $V$. Then the bounded cohomology $\mathrm{H}_{b}^{n}(G, V)$ of $G$ with coefficients in $V$ arises naturally by "quantifying" classes in ordinary cohomology.

The methods available to explicitly compute bounded cohomology are very sparse and very different from the ones used to compute ordinary group cohomology. There is not a single finitely generated group $G$, for which $\mathrm{H}_{b}^{n}(G, \mathbb{R})$ for all $n \geq 1$ is known, unless it is known to vanish in all degrees; see [Mon06].

In my research I gave several new constructions for bounded cohomology, including for the free group.
3.1. Cup Product in bounded cohomology. For a non-abelian free group $F$ it is known that $\mathrm{H}_{b}^{n}(F, \mathbb{R})$ is trivial if $n=1$ and uncountable dimensional in degrees $n=2,3$. For $n \geq 4$, it is unkown if $\mathrm{H}_{b}^{n}(F, \mathbb{R})$ is trivial. Free groups play a distinguished rôle in constructing non-trivial classes on other acylindrically hyperbolic groups: Any nontrivial alternating class in $\mathrm{H}_{b}^{n}(F, \mathbb{R})$ can be promoted to a non-trivial class in $\mathrm{H}_{b}^{n}(G, \mathbb{R})$ where $G$ is an acylindrically hyperbolic group and $n \geq 2$; see Corollary 2 of [FPS15]. Many groups of classical interest are known to be acylindrically hyperbolic, such as non-elementary hyperbolic groups, relatively hyperbolic groups, mapping class groups of punctured surfaces and $\operatorname{Out}\left(F_{n}\right)$ for $n \geq 2$.

Just like ordinary cohomology, $\mathrm{H}_{b}^{n}(G, \mathbb{R})$ has the structure of a graded ring via the cup product $\smile: \mathrm{H}_{b}^{n}(G, \mathbb{R}) \times \mathrm{H}_{b}^{m}(G, \mathbb{R}) \mapsto \mathrm{H}_{b}^{n+m}(G, \mathbb{R})$. In [Heu17] I could show that this cup product vanishes on all classical classes in $\mathrm{H}_{b}^{2}(F, \mathbb{R})$, where $F$ is a non-abelian free group.
Theorem F. [Heu17] Let $\alpha, \beta \in \mathrm{H}_{b}^{2}(F, \mathbb{R})$ be bounded classes which are induced by quasimorphisms of Brooks or Rolli ([Bro81], [Rol09]). Then

$$
\alpha \smile \beta \in \mathrm{H}_{b}^{4}(F, \mathbb{R})
$$

is trivial.
The quasimorphisms of Brooks are dense under pointwise convergence. However, one cannot deduce from this density that the above cup product always vanishes.
Question 3.1. Is the cup product $\smile: \mathrm{H}_{b}^{2}(F, \mathbb{R}) \times \mathrm{H}_{b}^{2}(F, \mathbb{R}) \rightarrow \mathrm{H}_{b}^{4}(F, \mathbb{R})$ trivial?
Note that it is unknown if $\mathrm{H}_{b}^{4}(F, \mathbb{R})$ is trivial.
3.2. Bounded Cohomology and Extensions of Groups. There is a well-known interpretation of degree 2 and 3 ordinary cohomology in terms of group extensions. A group extension of a group $G$ by a group $N$ is a group $E$ with a short exact sequence $1 \rightarrow N \rightarrow E \rightarrow G \rightarrow 1$ of groups. Every such group extension induces a homomorphism
$\psi: G \rightarrow \operatorname{Out}(N)$ by conjugation. Let $\mathcal{E}(G, N, \psi)$ be the set of all group extensions of $G$ by $N$ which induce $\psi$.

It is a well-known result of MacLane [Mac49] that this set is fully determined by the second and third ordinary group cohomology. In [Heub] I show that there is a analogous interpretation for bounded cohomology: Let $\mathcal{E}_{b}(G, N, \psi)$ be the set of extensions which have the additional property that there is a section $\sigma: G \rightarrow E$ which is a quasihomomorphism in the sense of Fujiwara and Kapovich ([FK16]) where ado $\sigma: G \rightarrow \operatorname{Aut}(N)$ has finite image.

Theorem G. [Heub] Let $G$ and $N$ be groups and suppose that $Z=Z(N)$, the centre of $N$, is equipped with a norm $\|\cdot\|$ such that $(Z,\|\cdot\|)$ has finite balls. Furthermore, let $\psi: G \rightarrow \operatorname{Out}(N)$ be a homomorphism with finite image.

There is a class $\omega_{b}=\omega_{b}(G, N, \psi) \in \mathrm{H}_{b}^{3}(G, Z)$ such that $\omega_{b}=0$ in $\mathrm{H}_{b}^{3}(G, Z)$ if and only if $\mathcal{E}_{b}(G, N, \psi) \neq \emptyset$ and $c^{3}\left(\omega_{b}\right)=0 \in \mathrm{H}^{3}(G, Z)$ and if and only if $\mathcal{E}(G, N, \psi) \neq \emptyset$. If $\mathcal{E}_{b}(G, N, \psi) \neq \emptyset$, there is a bijection between the sets $\mathrm{H}^{2}(G, Z)$ and $\mathcal{E}(G, N, \psi)$ which restricts to a bijection between $\operatorname{im}\left(c^{2}\right) \subset \mathrm{H}^{2}(G, Z)$ and $\mathcal{E}_{b}(G, N, \psi) \subset \mathcal{E}(G, N, \psi)$.

I could moreover classify the classes which arise as such an obstruction $\omega_{b} \in \mathrm{H}_{b}^{3}(G, Z)$.

## 4. Quasi-BNS invariants

BNS invariants were introduced by Bieri-Neumann-Strebel in [BNS87] to study the finiteness properties of normal subgroups containing the commutator subgroup. Let $G$ be a group with finite generating set $S$ and let $\operatorname{Cay}(G, S)$ be the Cayley graph of $G$ with respect to $S$. Then define the $B N S$-invariant $\Sigma^{1}(G) \subset \operatorname{Hom}(G, \mathbb{R})$ by setting

$$
\Sigma^{1}(G)=\left\{\phi \in \operatorname{Hom}(G, \mathbb{R}) \mid G_{\phi} \subset \operatorname{Cay}(G, S) \text { is connected }\right\}
$$

where $G_{\phi}=\{g \in G \mid \phi(g)>0\}$. The BNS invariants have been computed and exploited for many classes of groups. If $G$ is perfect the invariants are empty. On the other hand, there are many groups of interest for Geometric Group Theory that have a relative abundance of quasimorphisms. In joint work with Dawid Kielak I develop a theory of BNS-invariants based on quasimorphisms $G \rightarrow \mathbb{Z}$. For many perfect groups the resulting invariants are non-trivial and a rich theory is beginning to emerge.

## 5. Computational Complexity Commutator Length

The stable commutator length on non-abelian free groups is computable in polynomial time; [Cal09]. On the other hand the algorithms known to compute commutator length run in exponential time; see [Cul81].

In a joint project with Michał Marcinkowsk we study the computational complexity of commutator length in free groups. To be precise, we consider the decision problem with input $w \in F^{\prime}$ and $n \in \mathbb{N}$ which asks if $\operatorname{cl}(w) \leq n$. As $\operatorname{cl}(w) \leq n$ is verifiable by an explicit representation of $w$ as the product of $n$ or less commutators, this decision problem lies in $N P$.

Conjecture 5.1. The computation of commutator length in free groups is NP-complete.

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