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CHAPTER 1

STABLE COMMUTATOR LENGTH

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What is . . . stable commutator length? It is a real invariant which – just like any good invariant – has several incarnations within mathematics. Indeed, I will give three different equivalent definitions of stable commutator length (from here on scl), with some having a more topological, some a more algebraic and some a more analytic flavour.

This allows us to interconnect several mathematical invariants and fields. We shall see, for example, how invariants from computer science, dynamical systems, and graph theory may be used to construct interesting simplicial volumes – via scl, of course.

The aim of this article is to give a curated overview over the different areas in stable commutator length, mostly due to personal taste[1]. The question "What is . . . scl?" has been answered before by none less than Danny Calegari himself both in a short survey [**Cal08**] and in an extensive monograph [**Cal09a**]. The latter is the main reference for this introduction.

[1]and limited knowledge!

1. Three ways to stumble upon scl

There are three different categories in which to stumble upon scl: Topologically, algebraically and analytically, yielding three different definitions of scl.

In every case we will use the definition to compute the same invariant: The stable commutator length of a commutator in the free group.

1.1. scl **via Commutators.** Let G be a group. Recall that a commutator is the expression $[x, y] = xyx^{-1}y^{-1}$ for $x, y \in G$. The group generated by all commutators is called the *commutator subgroup*, denoted by $[G, G]$. For an element $g \in [G, G]$ the *commutator length* $\mathrm{cl}_G(g)$ measures how many commutators are needed to realise g as a product, i.e.

$$\mathrm{cl}_G(g) = \min\{n \mid g = [x_1, y_1] \cdots [x_n, y_n]; x_i, y_i \in G\}$$

Definition 1.1 (scl algebraically). Let $g \in G$ be an element in a group. If $g \in [G, G]$ lies in the commutator subgroup, we define the stable commutator length of g in G as

$$\mathrm{scl}_G(g) \coloneqq \lim_{n\to\infty} \frac{\mathrm{cl}_G(g^n)}{n}.$$

If some power g^N lies in the commutator subgroup, we set $\mathrm{scl}_G(g) = \frac{\mathrm{scl}_G(g^N)}{N}$. If no power of g lies in the commutator subgroup we set $\mathrm{scl}_G(g) = \infty$.

Commutator length is easily seen to be subadditive and thus the defining limit exist.

Example 1.2 (Commutators in Free group). Let $F = F(\{\mathtt{a}, \mathtt{b}\})$ be the free group with free generating set $\{\mathtt{a}, \mathtt{b}\}$. Then $\mathrm{cl}_F([\mathtt{a}, \mathtt{b}]) = 1$ (as $[\mathtt{a}, \mathtt{b}]$ is a commutator), $\mathrm{cl}_F([\mathtt{a}, \mathtt{b}]^2) = 2$ (as $[\mathtt{a}, \mathtt{b}]^2$ is not a commutator) and $\mathrm{cl}_F([\mathtt{a}, \mathtt{b}]^3) = 2$ (as scl is surprising and interesting [2]).

More generally, Culler [Cul81] found that $\mathrm{cl}_F([\mathtt{a}, \mathtt{b}]^n) = \lfloor \frac{n}{2} \rfloor + 1$. Taking the limit we see that

$$\mathrm{scl}_F([\mathtt{a}, \mathtt{b}]) = \frac{1}{2}.$$

1.2. scl **via Surfaces.** Let X be a topological space with a loop $\gamma\colon S^1 \to X$. A natural measure for the complexity of γ is the complexity of a surface Σ needed to *fill* γ.

So what do we mean by *filling* a loop? We mean that there is a map $\Phi\colon \Sigma \to X$ such that the boundary $\partial\Sigma$ maps to X and factors through γ via a map $\partial\Phi\colon \partial\Sigma \to$

[2] If you don't believe me, here is one way to see this: $[\mathtt{a}, \mathtt{b}]^3 = [\mathtt{aBA}, \mathtt{a}^2\mathtt{BAb}][\mathtt{BAb}, \mathtt{B}^2]$, where $\mathtt{A} = \mathtt{a}^{-1}$ and $\mathtt{B} = \mathtt{b}^{-1}$

S^1, i.e. such that the diagram

$$\begin{tikzcd} \partial\Sigma \arrow[r, "i"] \arrow[d, "\partial\Phi"] & \Sigma \arrow[d, "\Phi"] \\ S^1 \arrow[r, "\gamma"] & X \end{tikzcd}$$

commutes. The degree of $\partial\Phi$ will be denoted by $n(\Sigma, \Phi)$. We will call such surfaces *admissible*.

By *complexity* we mean - of course - the Euler characteristic of the surface, except that we only consider non-spherical components. I.e. we define $\chi^-(\Sigma) = \sum_{i=1}^n \min\{0, \chi(\Sigma_i)\}$, where Σ_i are the connected components of Σ. Finally we define:

DEFINITION 1.3 (scl topologically). Let $\gamma \colon S^1 \to X$ be a loop in a topological space X. If there is no admissible surface to γ we set $\mathrm{scl}_X(\gamma) = \infty$. Else, we set

$$\mathrm{scl}_X(\sum \gamma) := \inf_\Sigma \frac{1}{2} \cdot \frac{-\chi^-(\Sigma)}{n(\Sigma, \Phi)}$$

where the infimum ranges over all admissible surfaces with non-zero degree $n(\Sigma, \Phi)$.

EXAMPLE 1.4. Consider as a topological space X a torus with a disk removed. Let $\gamma \colon S^1 \to X$ be the boundary loop of that disk. Then, the space X itself is a surface Σ with boundary and the identity map has degree 1. Note that Σ has genus one and one boundary component and thus $\chi^-(\Sigma) = -1$. We thus estimate:

$$\mathrm{scl}_X(\gamma) \leq \frac{1}{2} \cdot \frac{-\chi^-(\Sigma)}{n(\Sigma, id)} = \frac{1}{2}.$$

We note that the fundamental group of X is the free group on two generators $\mathtt{a}$ and $\mathtt{b}$ (for example the meridian and longitude) and that γ corresponds to the commutator $[\mathtt{a}, \mathtt{b}]$ in the fundamental group.

1.3. scl **via Quasimorphisms.** Let G be a group and let $g \in G$ be an element. One of the few objects one may compute (in the Turing sense) from a given presentation are its homomorphisms $G \to \mathbb{R}$. This, however, is rather limiting: We are not able to *see* any element $g \in [G, G]$ in its abelian quotients.

We could of course ask for different target groups, for example by studying G via its finite quotients. We will take a slightly different approach and instead generalize the type of morphisms to $\mathbb{R}$. This is one way to motivate the following:

DEFINITION 1.5 (Quasimorphisms). A *quasimorphism* is a map $\phi \colon G \to \mathbb{R}$ such that there is a constant D, such that for all $g, h \in G$, $|\phi(g) + \phi(h) - \phi(gh)| \leq D$. The infimum of all such D is called the *defect* of ϕ and denoted by $D(\phi)$.

Observe that a map ϕ is a quasimorphism with $D(\phi) = 0$ if and only if it is a homomorphism. Quasimorphisms form a vector space under pointwise addition and scalar multiplication. The space of such quasimorphisms is enormous: Indeed, any bounded function is a quasimorphism. We may get rid of those by only considering *homogeneous* quasimorphisms, i.e. those quasimorphisms which additionally satisfy that $\phi(g^n) = n \cdot \phi(g)$ for all $n \in \mathbb{Z}$ and $g \in G$. It may be seen[3] that every quasimorphism has a unique homogeneous quasimorphism in bounded distance. Homogeneous quasimorphisms have some interesting properties, for example $\phi(g) = \phi(hgh^{-1})$ for all $g, h \in G$.

Fix an element $g \in [G, G]$. We may see that for any quasimorphism $\phi(g)$, we may *bound* $\phi(g)$ uniformly in terms of $D(\phi)$: For example for any commutator $[x, y]$ we can write $|\phi([x, y])| \leq |\phi(x) + \phi(y) + \phi(x^{-1}) + \phi(y^{-1})| + 3D(\phi) \leq 5D(\phi)$ by successively applying the definition of quasimorphisms. On the other hand, if $g \in G$ is such that there is any homomorphism $\phi \colon G \to \mathbb{R}$ with $\phi(g) > 0$ then we may never bound g just in terms of $D(\phi)$, since we may arbitrarily scale up ϕ. We thus consider the following:

Definition 1.6 (scl via Quasimorphisms; [**Bav91**]). Let G be a group and let $g \in G$ be an element. Then

$$\mathrm{scl}_G^{qm}(g) := \sup_{\phi} \frac{1}{2} \cdot \frac{\phi(g)}{D(\phi)},$$

where the supremum ranges over all homogeneous quasimorphisms $\phi \colon G \to \mathbb{R}$ with defect $D(\phi)$.

Example 1.7 (Brooks Quasimorphisms). Let $F = F(\{\mathtt{a}, \mathtt{b}\})$ be the non-abelian free group with free generating set $\{\mathtt{a}, \mathtt{b}\}$. We will also write $\mathtt{A}$ for $\mathtt{a}^{-1}$ and $\mathtt{B}$ for $\mathtt{b}^{-1}$. Fix a word $x \in F$. We denote by $\nu_x \colon F \to \mathbb{N}$ the map that associates to a word w the number largest number of times x is a subword of w, i.e. the maximum of all n such that

$$w = w_0 \cdot x \cdot w_1 \cdots w_{n-1} \cdot x \cdot w_n$$

for appropriate w_i and where this expression is reduced. We define $\phi_x \colon F \to \mathbb{Z}$ via $\phi_x(w) \mapsto \nu_x(w) - \nu_{x^{-1}}(w)$. It turns out that ϕ_x is a quasimorphism, called *Brooks quasimorphism* with $D(\phi_x) \leq 3$. Those maps were originally introduced by Brooks [**Bro81**] to show that the space of quasimorphisms on the free group is infinite dimensional.

Consider the element $[\mathtt{a}, \mathtt{b}] \in F$. It may be seen [**Heu19c**, Section 2.4] that $\phi := \phi_{\mathtt{ab}} - \phi_{\mathtt{ba}}$ satisfies $D(\phi) = 1$ and the associated homogeneous quasimorphism $\bar{\phi}$ satisfies $D(\bar{\phi}) = 2$. Moreover one may compute that $\bar{\phi}([\mathtt{a}, \mathtt{b}]) = 2$.

[3]Consider for a quasimorphism $\phi \colon G \to \mathbb{R}$ the map $\bar{\phi}(g) = \lim_{n\to\infty} \frac{\phi(g^n)}{n}$, called the *homogenization of* ϕ. We have that $\bar{\phi}$ is a quasimorphism and $D(\bar{\phi}) \leq 2 \cdot D(\phi)$ [**Cal09a**, Lemma 2.21]

Putting things together we estimate:

$$\mathrm{scl}_F^{qm}([\mathtt{a},\mathtt{b}]) \geq \frac{1}{2} \cdot \frac{\bar{\phi}([\mathtt{a},\mathtt{b}])}{D(\bar{\phi})} = \frac{1}{2}.$$

Quasimorphisms (and thus stable commutator length) are directly related to bounded cohomology as follows.

Proposition 1.8. *Let G be a group. The vectorspace of quasimorphisms $G \to \mathbb{R}$ modulus the vector space of trivial quasimorphisms is in a 1-1 correspondence to $\ker(c^2)$, the kernel of the comparison map $c\colon \mathrm{H}_b^\bullet(G;\mathbb{R}) \to \mathrm{H}^\bullet(G;\mathbb{R})$.*

1.4. Wrapping things up: Equivalence of definitions. Of course, all of the Definitions 1.3, 1.1 and 1.6 of stable commutator length are equivalent!

Theorem 1.9 (Calegari [**Cal09a**, Proposition 2.10] and Bavard [**Bav91**]). *Let X be a topological space with fundamental group G and let $\gamma\colon S^1 \to X$ be a loop corresponding to the element $g \in G$. Then*

$$\mathrm{scl}_X(\gamma) = \mathrm{scl}_G(g) = \mathrm{scl}_G^{qm}(g).$$

Thus Examples 1.2, 1.4 and 1.7 computed and estimated exactly the same fact, namely that the stable commutator length of a commutator in the non-abelian free group on two generators is $\frac{1}{2}$. Note the different nature of the invariants involved: scl_X is an infimum, scl_G is a limit and scl_G^{qm} is a supremum. It turns out that the infimum in scl_X is only sometimes achieved, while the supremum in scl_G^{qm} is always achieved. The quasimorphisms which achieve this supremum are also called *extremal quasimorphisms*.

Since all definitions are equivalent we will now only study scl on groups. We also note that scl generalizes to formal sums of elements, called *chains*.

We collect some basic properties:

Proposition 1.10 (Monotonicity). *For any homomorphism $\Phi\colon G \to H$ between two groups we have that $\mathrm{scl}_G(g) \geq \mathrm{scl}_H(\Phi(g))$. From this we see that* scl *is invariant under automorphisms, and invariant under retractions.*

Proposition 1.11 (Finite Index Subgroups). *If $H < G$ is a normal finite index subgroup and $g \in H$, then we may compute $\mathrm{scl}_H(h)$ in terms of scl_G as follows:*

$$\mathrm{scl}_H(h) = \mathrm{scl}_G(\sum_{a \in A} \mathrm{scl}_G(aha^{-1})$$

for $A = G/H$ and where $a \in A$ is any representative.

2. Vanishing, Gaps and Lions

So: What is stable commutator length? After having seen three equivalent definitions we will explore scl on finitely presented groups.

Martin Bridson [**Bri06**, Figure 1] charted the landscape of finitely presented groups by means of their large scale geometry. Starting from $\mathbb{Z}$, one may explore finitely presented groups in two very different directions: One may follow the amenable path, passing by (in increasing difficulty) the abelian, nilpotent, polycyclic and solvable groups. One may also wander off in the directions of groups of negative (or non-positive) curvature, passing by free, hyperbolic, semi-hyperbolic, CAT(0) and acylindrically hyperbolic groups.

The limits of exploring finitely presented groups are embodied by Lions: As the workings of any Turing machine may be encoded in a finitely presented group, and as basic questions of Turing machines are undecidable, there is no hope of understanding or computing meaningful invariants from arbitrary finitely presented groups.

We will see that scl respects this landscape: Either scl vanishes on the whole group (in the amenable case), or scl of a group may be uniformly bounded from below (for groups with non-positive curvature). And we will see that also scl can not tame the lions, though there has been some progress by cornering them to *right-computable numbers* (Theorem 3.3).

2.1. Vanishing. Stable commutator length vanishes for any group with trivial real second bounded cohomology. This implies that $\mathrm{scl}_G(g) = 0$ for any amenable group G and $g \in G$. This is a huge class of groups, encompassing, for example, all solvable groups. Besides this some other vanishing results are known, for example for subgroup of piecewise linear transformations of the interval [**Cal07**].

2.2. Gaps. In contrast, many classes of non-positively curved groups have a *gap* in stable commutator length. A group G is said to have such a *spectral gap*, if there is a constant $C > 0$ such that for any element $g \in G$ either $\mathrm{scl}_G(g) = 0$ or $\mathrm{scl}_G(g) \geq C$. The largest C is called the *optimal gap* and denoted by C_G. Typically, one may also control the elements which satisfy $\mathrm{scl}_G(g) = 0$ and we say that G has a *strong gap* if the only element satisfying $\mathrm{scl}_G(g) = 0$ is the identity. We will see that all groups satisfy $C_G \leq \frac{1}{2}$.

Why might such a gap be useful? It allows us to obstruct and bound subgroups as follows: Suppose that $H < G$ is a subgroup of G and that G has a strong scl gap C_G. It follows from the monotonicity (Proposition 1.10) that also H has a strong scl gap and that $C_H \geq C_G$. Thus, a group G with a strong gap of $C_G = \frac{1}{2}$ only have subgroups with strong gaps and $C_H = \frac{1}{2}$. This may be seen as some crude algebraic Tits-alternative. Moreover, if $H < G$ is a finite index subgroup, and H has a gap $\tilde{C}_H$ for *chains*, by the index formula (Proposition 1.11) we may estimate that $\tilde{C}_H \leq \frac{1}{[G:H]} \tilde{C}_G$. This allows one to estimate the indices of subgroups.

We list some notable results on scl gaps:

THEOREM 2.1. *We have spectral gaps in the following cases:*

(1) *Any Gromov hyperbolic group* [CF10, Theorem A], *though this gap is not uniform. An element g has $\mathrm{scl}_G(g) = 0$ if and only if g^n is conjugate to g^{-n} for some $n \in \mathbb{Z}_+$.*
(2) *Any finite index subgroup of the mapping class group* $\mathrm{Mod}(\Sigma)$ *of a possibly punctured closed orientable surface* Σ [BBF16, Theorem B]*) There is a similar characterization for elements with vanishing* scl.
(3) *The fundamental group of any* 3*-manifold group* [CH19, Theorem C]
(4) *Any (subgroup of a) RAAG, in particular any special group, even a strong gap of precisely* $\frac{1}{2}$ [Heu19c]*. See also* [FFT19] *and* [FST20].
(5) *Elements in free and certain amalgamated free products which do not conjugate into vertex groups* [CFL16, Che18]
(6) *Elements in graph products which do not conjugate into vertex groups* [CH20b]

2.3. Lions. Beyond the realms of amenability and hyperbolicity lie the Lions, the groups impossible to slay by means of Turing machines. Given a finitely presented group and an element $g \in G$ it is undecidable if $\mathrm{scl}_G(g) = 0$ or even if $\mathrm{scl}_G(g) \leq C$ for any real number C.[4] However, we may somewhat corner the Lions (i.e. arbitrary finitely presented groups): We will see (Theorem 3.3) that the scls of finitely presented groups are always right-computable.

3. Spectrum

We now explore the spectrum of scl for a given group or class of groups, i.e. the set $\mathrm{scl}_G(G) \subset \mathbb{R}_{\geq 0}$. We will start with the free group. Calegari [Cal09b] found an algorithm[5] to compute scl on free groups. The algorithm showed that scl is rational on free groups. The algorithm also allowed for computer experiments on the distribution of scl of random elements, which revealed a striking distribution; see Figure 1.[6] I emphasize that the only thing known about this figure is that scl $\geq 1/2$ and scl is rational - that's it! By merely looking at this figure we may make two educated guesses (if not conjectures): The spectrum gets very sparse to the left, i.e. there seems to be a second gap in scl[7] , and scl seems to be much more frequent

[4]Here is one way to see this: For a finitely presented group G and an element $g \in G$ in it we will construct an element $\tilde{g}$ in a group $\tilde{G}$ such that $\mathrm{scl}_{\tilde{G}}(\tilde{g}) = 0$ if and only if g is trivial in G and $\mathrm{scl}_{\tilde{G}}(\tilde{g}) = \frac{1}{2}$, else. We may assume that g is infinite torsion by replacing g by $g_1 g_2 \in G_1 \star G_2$ in $G_1 \star G_2$, where both g_i and G_i for $i \in \{1, 2\}$ are a copy of g and G respectively. We may then observe that $\tilde{g} = [g, t] \in G \star \langle t \rangle \coloneqq \tilde{G}$ is trivial if and only if g is trivial in G and that $\mathrm{scl}_{\tilde{G}}(\tilde{g}) = \frac{1}{2}$, using the work of Chen [Che18]

[5]by the name of `scallop`, which has been implemented by Alden Walker [CW09]. It may be downloaded on Github, is very fast, and very interesting to play around with!

[6]The dataset of the 50.000 random scls and the (Python) code to generate this figure may be found at `nicolausheuer.com/code.html`

[7]formally: there seems to be no $g \in F$ such that $1/2 < \mathrm{scl}_F(g) < 7/12$

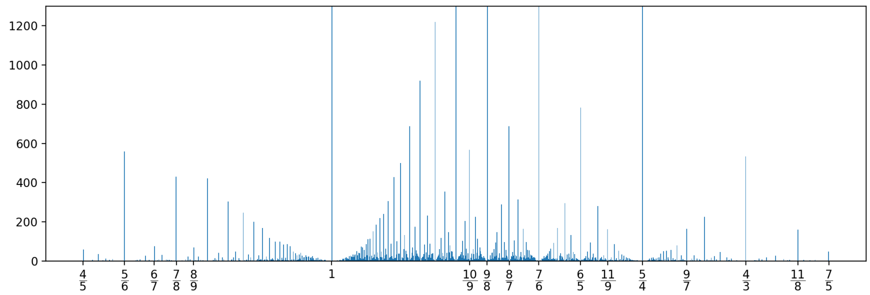

FIGURE 1. Histogram of scl of 50'000 random words of length 24 in $[F_2, F_2]$ using `scallop` [**CW09**]

on elements with low denominator, in particular the frequency of $p/(2q)$ seems to be proportional to q^{-d} for $d \sim 2$. A statistical analysis of this phenomenon may be found in [**CH20b**, Section 7.3].

One may generalize rationality of scl to certain free products [**Che18**] and to certain amalgamated free products [**?, Che20**] including Baumslag–Solitair groups.

Besides free groups, very little is known about the spectrum of scl for other hyperbolic groups. It is unknown if scl is rational in surfaces groups, besides for certain elements [**FM00**].

Which other values may scl take? A big source of examples comes from circle actions. There is a well established connection between circle actions and bounded cohomology to do Ghys [**Ghy01**]. Given a group G with an action on the circle $\rho\colon G \to \mathrm{Homeo}^+(S^1)$. This action allows us to cyclically extend G via the Euler class associated to ρ[8] to a group $\tilde{G}$. Then the action ρ on G lifts to an action $\tilde{\rho}\colon G \to \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$, the group of orientation preserving homomorphisms $\phi\colon \mathbb{R} \to \mathbb{R}$ of the reals which commute with the integers, i.e. such that $\phi(x+n) = \phi(x) + n$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$. We define the *rotation number* of such ϕ via:

$$\mathrm{rot}\colon \phi \to \lim_{n\to\infty} \frac{\phi^n(0)}{n}.$$

A key insight is that $\mathrm{rot}\colon \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R}) \to \mathbb{R}$ is a homogeneous quasimorphism of defect 1, which hence defines a quasimorphism on G by pulling rot back via $\tilde{\rho}$. We have:

THEOREM 3.1 (scl and rotation number; [**Cal09a**, Section 5]). *Let G be a perfect group which satisfies $\mathrm{scl}_G(G) = 0$ and which admits a non-trivial action $\rho\colon G \to S^1$ on the circle. Then for any element $\tilde{g} \in \tilde{G}$ in the central extension of G associated*

[8]A introduction to this may be found in [**BFH14**]

to the Euler class of ρ we have that

$$\mathrm{scl}_{\tilde{G}} \colon \tilde{g} \mapsto \frac{|\mathrm{rot}(\tilde{g})|}{2},$$

where rot *is the rotation number of $\tilde{g}$.*

Note that this already shows that $\mathrm{scl}_{\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})}(\mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})) = \mathbb{R}_{\geq 0}$. Rotation number has been well studied for several groups acting on the circle. An interesting example of groups with vanishing scl is the group of piecewise linear transformations of the interval as seen in Section 2.

For example using Thompson's Group T in Theorem 3.1 one constructs a finitely presented group whose scl spectrum is *exactly* $\mathbb{Q}_{\geq 0}$ [**Cal09a**, Remark 5.20]. Zhuang used the Stein-Thompson's Groups to give the first example of finitely presented groups which have non-rational scl.

Theorem 3.2 ([**Zhu07**]). *There are finitely presented groups which have transcendental stable commutator lengths.*

All of the stable commutator lengths he constructed are a quotient of logarithms, e.g. $\log(3)/\log(2)$. Such numbers are either rational or transcendental. It is unknown if there are finitely presented groups which admit scls which are algebraic and not rational. I also mention that, using a connection to the fractional stability number of graphs, one may construct groups with exotic spectrum, such as groups which have a gap but are eventually dense [**CH20b**, Theorem I, J].

More is known by considering *recursively presented* groups. These are all finitely generated subgroups of finitely presented group. Note that the set of recursively presented groups is countable, and thus so is the set of scls on it. It is possibly to characterize the set of scls on this class of groups by their computability.

Theorem 3.3 ([**Heu19b**, Theorem A]). *The set of stable commutator length on recursively presented groups equals to the set of non-negative right-computable numbers.*

A non-negative real number α is called right-computable if there is a Turing machine T which for any $i \in \mathbb{N}$ returns a rational number $T(i) \geq 0$ such that $T(i+1) \leq T(i)$ for all $i \in \mathbb{N}$ and $\alpha = \lim_{i \to \infty} T(i)$.

4. Relationship to simplicial volume

One application of stable commutator length is to construct manifolds with controlled simplicial volume.

Theorem 4.1 ([**HL20b**, Theorem F]). *Let G be a finitely presented group such that $\mathrm{H}_2(G;\mathbb{R}) \cong 0$. Then for any $g \in G$ there is an orientable closed connected (occ)* *manifold M such that*

$$\|M\| = 48 \cdot \mathrm{scl}_G(g),$$

where $\|M\|$ denotes the simplicial volume of M.

We may use this theorem to translate the spectral results for scl known from Section 3 to the simplicial volume of manifolds. Using such techniques we see:

THEOREM 4.2 ([**HL20b**, **HL20c**, **HL20a**]).

(1) The set of simplicial volumes of occ n-manifolds is dense in $\mathbb{R}_{\geq 0}$ for all $n \geq 4$.
(2) Every rational is the simplicial volume of an occ 4-manifold. Moreover, there is a sequence M_i of occ 4-manifolds such that $\|M_i\| \to 0$ and such that $\|M_i\|$ are all linearly independent over the algebraic numbers and in particular transcendental.
(3) The set of locally finite simplicial volumes of oriented connected open n-manifolds is $\mathbb{R}_{\geq 0}$ for any $n \geq 4$.

5. Open Questions in scl

I end this article by listing some open questions about stable commutator length.

(1) What are extremal quasimorphisms for arbitrary elements of the free group?
(2) Is there a second gap of scl in non-abelian free groups F, i.e. are there no elements $g \in F$ such that $\frac{1}{2} \leq \mathrm{scl}_F(g) \leq \frac{7}{12}$?
(3) Is there a finitely presented group which has algebraic but not rational values scl? Is the set of scls on finitely presented groups the set of right-computable numbers?
(4) Is scl rational on surface groups? If yes, is this rationality achieved using extremal surfaces? What about scl on Gromov hyperbolic groups?

    This is, at least qualitatively, related to Gromov's Question: Does every one-ended hyperbolic group contain a surface subgroup?
(5) In the free group: Is there a connection to the primitivity rank in free groups? Recall that for an element $w \in F$ in a free group F the primitivity rank is defined as $\pi(w) = \min\{\mathrm{rk}(H)\}$ where $H < F$ runs over all subgroups of F such that $w \in H$ is *not* primitive in H. It was shown that the primitivity rank plays a crucial role in understanding the geometry of its associated one-relator subgroup [**LW18**]. It was conjectured [**Heu19a**, Conjecture 6.3.2] that for all $w \in F$, $\mathrm{scl}(w) \geq \frac{\pi(w)-1}{2}$. This would generalize the gap for elements in free groups. This conjecture has been verified for all words up to length 16 in free groups [**CH20a**].

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