Stable Commutator Length in Right Angled Artin Groups

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Background

Right-Anlged Artin Groups

For what follows, Γ will be a finite simplicial graph with vertices $\mathrm{V}(\Gamma)$ and edges $\mathrm{E}(\Gamma)$.

Definition

The right angled Artin group (RAAG) $\mathrm{A}(\Gamma)$ associated to Γ is defined via

$$\mathrm{A}(\Gamma) = \langle \mathrm{V}(\Gamma) \mid [v, w]; (v, w) \in \mathrm{E}(\Gamma) \rangle$$

Many groups are now known to be the subgroup or virtually the subgroups of RAAGs:

Example

- (Non-)abelian free groups, higher genus surface groups
- Fundamental groups of special cube complexes
- Fundamental groups of hyperbolic 3-manifolds

Stable Commutator Length

For $g, h \in G$, *commutator* $[g, h] \in G$ denotes the element $ghg^{-1}h^{-1}$. The *commutator subgroup* denotes the subgroup $G' < G$ generated by the commutators. For an element $g \in G'$ the *commutator length* ($\mathrm{cl}(g)$) denotes the word length with respect to this generating set.

Definition

Let $g \in G'$. The *stable commutator length* ($\mathrm{scl}(g)$) of g is defined via

$$\mathrm{scl}(g) = \lim_{n \to \infty} \frac{\mathrm{cl}(g^n)}{n}$$

Stable commutator length is *monotone* and *characteristic* i.e. for every homomorphism $\phi \colon G \to H$, $\mathrm{scl}(g) \geq \mathrm{scl}(\phi(g))$ and for every automorphism $\phi \colon G \to G$, $\mathrm{scl}(g) = \mathrm{scl}(\phi(g))$.
Stable commutator length encodes the complexity of surface maps to the classifying space. The theory of these invariants was developed by Calegari in [Cal09].

(Spectral) Gaps in scl

For a group G the *spectral gap* is the supremum over all reals $C \geq 0$ such that for any $e \neq g \in G'$, $\mathrm{scl}(g) \geq C$. Such a gap is necessarily bounded above by $1/2$. Many natural classes of groups have a positive spectral gap:

- Residually free gap have a gap of exactly $1/2$; see [DH91].
- Elements $g \in G_1 \star G_2$ in a free product where g does not conjugate into one of the factors; see [Che16].
- Hyperbolic groups have a gap, which depends on the hyperbolicity constant and the number of generators; see [CF10].
- Many other classes like Baumslag-Solitar groups, Mapping class groups, etc.

Quasimorphisms and Bavard's Dulaity Theorem

A *quasimorphism* $\phi \colon G \to \mathbb{R}$ is a map such that there is a $C > 0$, such that for every $g, h \in G$ $|\phi(g) + \phi(h) - \phi(gh)| \leq C$. The least such C is called the *defect of* ϕ and is denoted by $D(\phi)$. A quasimorphism ϕ is said to be *homogeneous* if for every $g \in G$, $n \in \mathbb{Z}$ we have that $\phi(g^n) = n\phi(g)$. Quasimorphisms may be used to compute scl using *Bavard's Duality Theorem*:

Theorem ([Bav91])

Let G be a group and let $g \in G'$. Then

$$\mathrm{scl}(g) = \sup_{\phi \in Q(G)} \frac{\phi(g)}{2D(\phi)}$$

where $Q(G)$ denotes the vectorspace of homogeneous quasimorphisms.

Left relatively convex subgroups

Definition

A subgroup $H < G$ is *left-relatively convex* if there is a G-invariant order $\prec$ on the right cosets.

In [ADS15] the authors studied left-relatively convex subgroups. They showed that

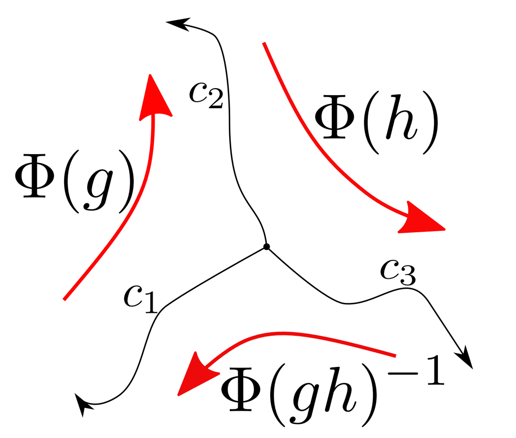
Theorem ([ADS15])

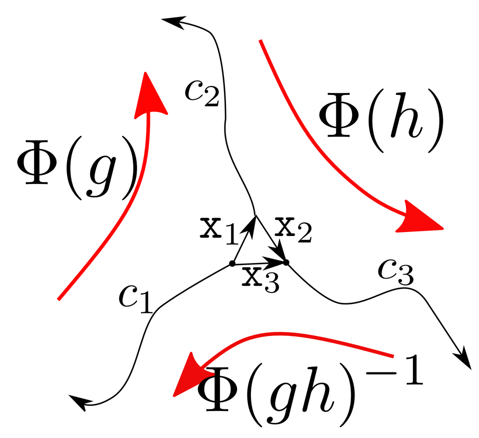
Let $\Lambda \subset \Gamma$ be a full subgraph of Γ. Then $\mathrm{A}(\Lambda) < \mathrm{A}(\Gamma)$ is left relatively convex.

Results

Homomorphisms vs. Letter Quasimorphisms

We want to generalise homomorphisms $\Phi \colon G \to \mathbb{F}_2$ since not all groups have 'enough' such maps. For what follows $\mathbb{F}_2 = \langle \mathtt{a}, \mathtt{b} \rangle$ denotes the free group on the letters $\mathtt{a}, \mathtt{b}$ and $\mathcal{A} \subset \mathbb{F}_2$ denotes the subset of *alternating words* i.e. words in which no higher powers of $\mathtt{a}, \mathtt{b}$ occur as subwords.

<table>
<tr><th>Homomorphism</th><th>Letter Quasimorphisms</th></tr>
<tr><td>A map $\Phi \colon G \to \mathbb{F}_2$ is a homomorphism if for every two elements $g, h \in G$, the elements $(\Phi(g), \Phi(h), \Phi(gh)^{-1})$ form a thin triangle in the Cayley graph: there are elements $c_1, c_2, c_3 \in \mathbb{F}_2$ such that
$$\Phi(g) = c_1 c_2^{-1}$$ $$\Phi(h) = c_2 c_3^{-1}$$ $$\Phi(gh) = c_1 c_3^{-1}$$
as reduced words in $\mathbb{F}_2$.</td><td>A map $\Phi \colon G \to \mathcal{A}$ is called letter-quasimorphism if for every two elements $g, h \in G$, the elements $(\Phi(g), \Phi(h), \Phi(gh)^{-1})$ almost form a 'thin triangle': there are elements $c_1, c_2, c_3 \in \mathbb{F}_2$ and letters $\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3$ such that
$$\Phi(g) = c_1 \mathtt{x}_1 c_2^{-1}$$ $$\Phi(h) = c_2 \mathtt{x}_2 c_3^{-1}$$ $$\Phi(gh) = c_1 \mathtt{x}_3 c_3^{-1}$$
as reduced words in $\mathcal{A}$.
We additionally require that $\mathtt{x}_i$ are letters of the same type and that $\mathtt{x}_1 \mathtt{x}_2 \mathtt{x}_3$ is a letter as well.</td></tr>
<tr><td>The corresponding picture is:


This triangle lies in the Cayley Graph.</td><td>The corresponding picture is:


This 'triangle' does not lie in any Cayley Graph.</td></tr>
</table>

Letter-Quasimorphisms and Spectral Gaps

If $g \in G'$ is a non-trivial element and $\Phi \colon G \to \mathbb{F}_2$ is a homomorphism such that $\Phi(g)$ is non-trivial, then $\mathrm{scl}(g) \geq 1/2$ by monotinicity of scl and since $\mathbb{F}_2$ has a scl-gap of $1/2$. Similarly:

Theorem (H. '18, [Heu18])

Let $g \in G$ be an element and let $\Phi \colon G \to \mathcal{A}$ be a letter-quasimorphism such that $\Phi(g^n) = \Phi(g)^n$ for $n \in \mathbb{N}$. Then there is an explicit homogeneous quasimorphism $\phi \colon G \to \mathbb{R}$ such that $\phi(g) \geq 1$ and $D(\phi) = 1$. By Bavard's Duality Theorem, $\mathrm{scl}(g) \geq 1/2$.

Letter quasimorphisms arise naturally under the presence of left-invariant orders and left-invariant subgroups.

Example

Let $\Phi \colon \mathbb{F}_2 = \langle \mathtt{a}, \mathtt{b} \rangle \to \mathcal{A}$ be the map defined via

$$\Phi \colon \mathtt{a}^{n_1}\mathtt{b}^{m_1} \cdots \mathtt{a}^{n_k}\mathtt{b}^{n_k} \mapsto \mathtt{a}^{\mathrm{sign}(n_1)}\mathtt{b}^{\mathrm{sign}(m_1)} \cdots \mathtt{a}^{\mathrm{sign}(n_k)}\mathtt{b}^{\mathrm{sign}(n_k)}$$

then Φ is a letter quasimorphism.

Spectral Gaps in RAAGs and amalgamated free products

Generalising the previous example we may prove:

Theorem (H. '18, [Heu18])

Let $G = A \star_C B$ be an amalgamated free product over a group C which embeds left relatively convex in A and B. Then every element $g \in G'$ which does not conjugate into one of the factors satisfies $\mathrm{scl}(g) \geq 1/2$.

Realising RAAGs as amalgamations of a star over a vertex with the complement over the link we may show:

Theorem (H. '18, [Heu18])

Every element $g \in G'$ in the commutator subgroup of a right-angled Artin group G satisfies $\mathrm{scl}(g) \geq 1/2$. This bound is sharp.

This is an improvement of a bound previously found by [FFT16] and [FST17].

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