The spectrum of simplicial volume

Nicolaus Heuer, University of Cambridge
joint with
Clara Löh, University of Regensburg

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Motivation of Simplicial Volume

Let M be a (topological) manifold. What is a good notion for $\boldsymbol{Vol(M)}$?

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Volume wish-list:

1. $Vol(M) \in \mathbb{R} \geq 0$
2. If M and N are *n* manifolds and $f: M \to N$ is continuous, then $|\deg(M)|\, Vol(N) \leq Vol(M)$
3. $Vol(M\#N) = Vol(M) + Vol(N)$
4. $Vol(M \times N) = Vol(M) \cdot Vol(N)$
5. $Vol(M)$ reflects honest Riemannian volume in special cases

Definition Simplicial Volume

Def (Gromov, 1984): If M is an orientable, closed, connected (occ) n-manifold. Define

$$||M||:= ||[M]||_1$$

the *simplicial volume,* where $[M]$ is the fundamental class of M.

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Def: Let X be a topological space and let $\alpha \in Hn(X;\ \mathbb{R})$ be a class in singular homology. Then set

$$||\alpha||_1 \coloneqq \inf\{ \sum|\alpha\sigma |;\ \alpha = \sum\alpha\sigma \cdot \sigma \}$$

the l^1 *norm* of α.

Properties of Simplicial Volume

$$\boldsymbol{Vol(M) = ||M||?}$$

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4. $||M \times N|| = ||M|| \cdot ||N||$ ❌

$$||M|| \cdot ||N|| \leq \left||M \times N\right|| \leq \binom{n+m}{n}\ ||M|| \cdot ||N||$$

5. $||M||$ reflects honest Riemannian volume in special cases ✅

- M hyperbolic: $||M|| = \frac{Vol(M)}{c_n}$, c_n only depends on dimension of M
- M amenable: $||M|| = 0$.
- $\frac{||M||}{(n-1)^n \cdot n!} \leq minVol(M)$, n: dimension of M.

Spectrum of Simplicial Volume

Characterize

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A priori:

- $SV(n)$ is countable
- if $a, b \in SV(n)$ then $a + b \in SV(n)$

$$SV(n) = \{ ||M|| \mid M \ is \ occ \ n - manifold \}.$$

Dimension Simplicial Volumes

$n = 2$

- Sphere
- Torus
- Higher genus surface

$$SV(n) = \{ ||M|| \mid M \text{ is occ } n - manifold\}.$$

Dimension Simplicial Volumes

$n = 2$ $SV(2) = \mathbb{N}[4]$

- Sphere $||S^2||$ = 0
- Torus $||T^2||$ = 0
- Higher genus surface $||\Sigma_g||$ = $-2\,\chi(\Sigma_g) = 4\,g - 4$

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- Products of surfaces?

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- Products of surfaces? Smallest product is 24

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Is there a gap in $SV(4)$?

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$n = 4$	**Theorem A** (H. – Löh, 2020, [HLa, HLb]): $SV(4)$ contains all non-negative rational numbers and arbitrarily small transcendental, rationally independent numbers.
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Is $SV(4) = RC^+$ the set of non-negative right computable numbers?

Outline: Proof of Theorem A $(SV(4) \supseteq \mathbb{Q}^{+})$

1. Construct 2-classes with l^1 norm controlled by stable commutator length
2. Find groups with interesting stable commutator length
3. From 2-classes to 4-classes using products
4. From 4-classes in groups to manifolds: Thom Realisation

Proof of Theorem A:
1. Construct 2-classes with l^1 norm controlled by stable commutator length

Let X be a topological space

	$\boldsymbol{l1}$ –norm	scl
Objects	$\alpha \in H_2(X;\mathbb{R})$	
Approximation by Surfaces		
Example		

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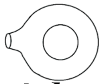
Example	$X = \Sigma_2, \quad \alpha = [\Sigma_2]$ $\Phi = id : \Sigma_2 \to \mathrm{X}$ $\lvert\lvert\alpha\rvert\rvert \leq \frac{-2\chi(\Sigma_2)}{n(\Phi)} = 4$	$X = \Sigma_{1,1} =$  $, \quad \gamma = \partial\Sigma_{1,1}$ $\Phi = id : \Sigma_{1,1} \to \mathrm{X}$ $scl(\gamma) \leq -\frac{\chi(\Sigma_{1,1})}{2\, n(\Phi)} = \frac{1}{2}$

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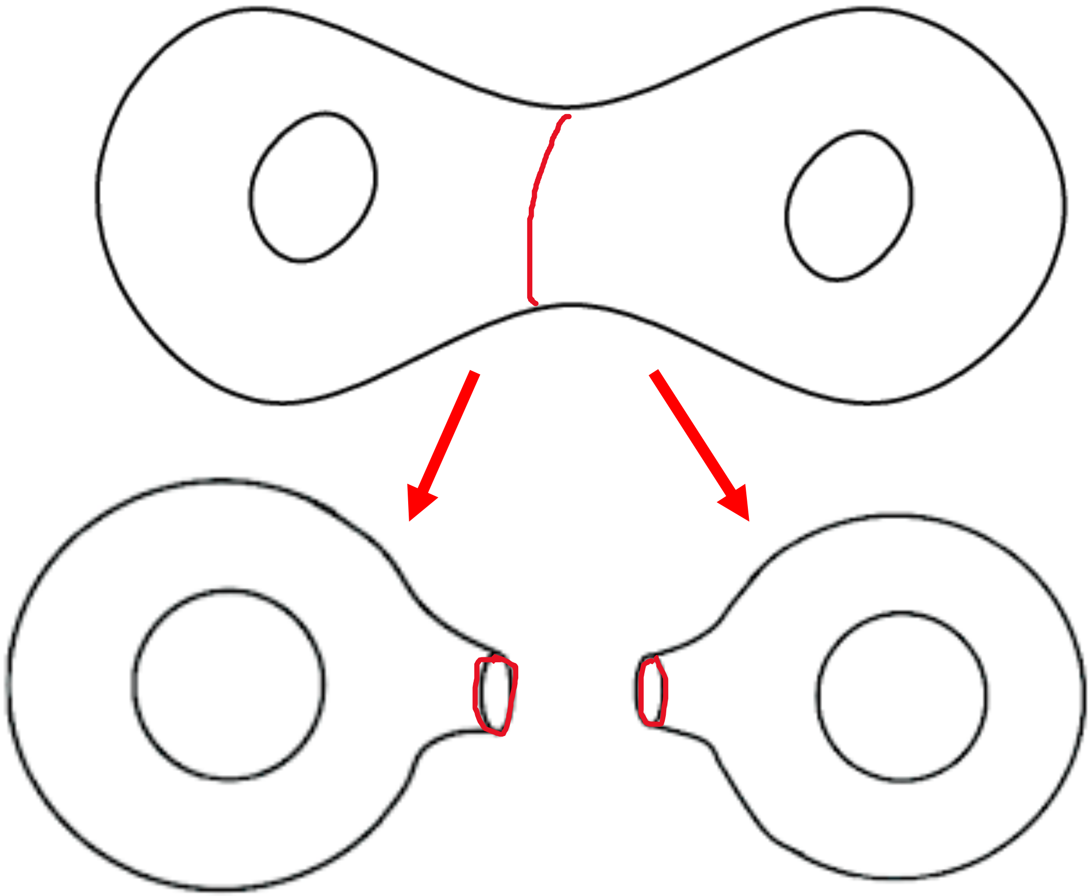

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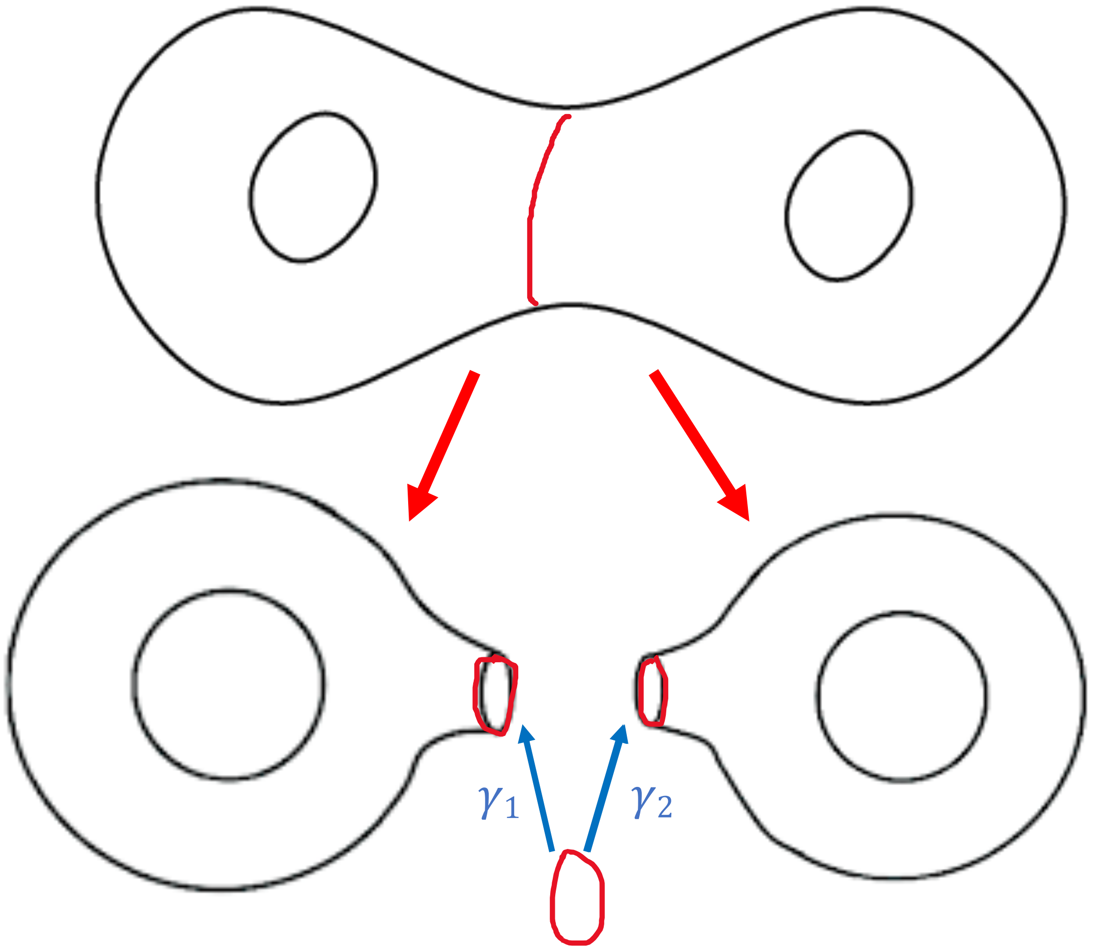

$$\|\Sigma_2\|$$

$$=$$

$$4 \cdot scl(\gamma_1) + 4 \cdot scl(\gamma_2)$$

$$4 = 4 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2}$$

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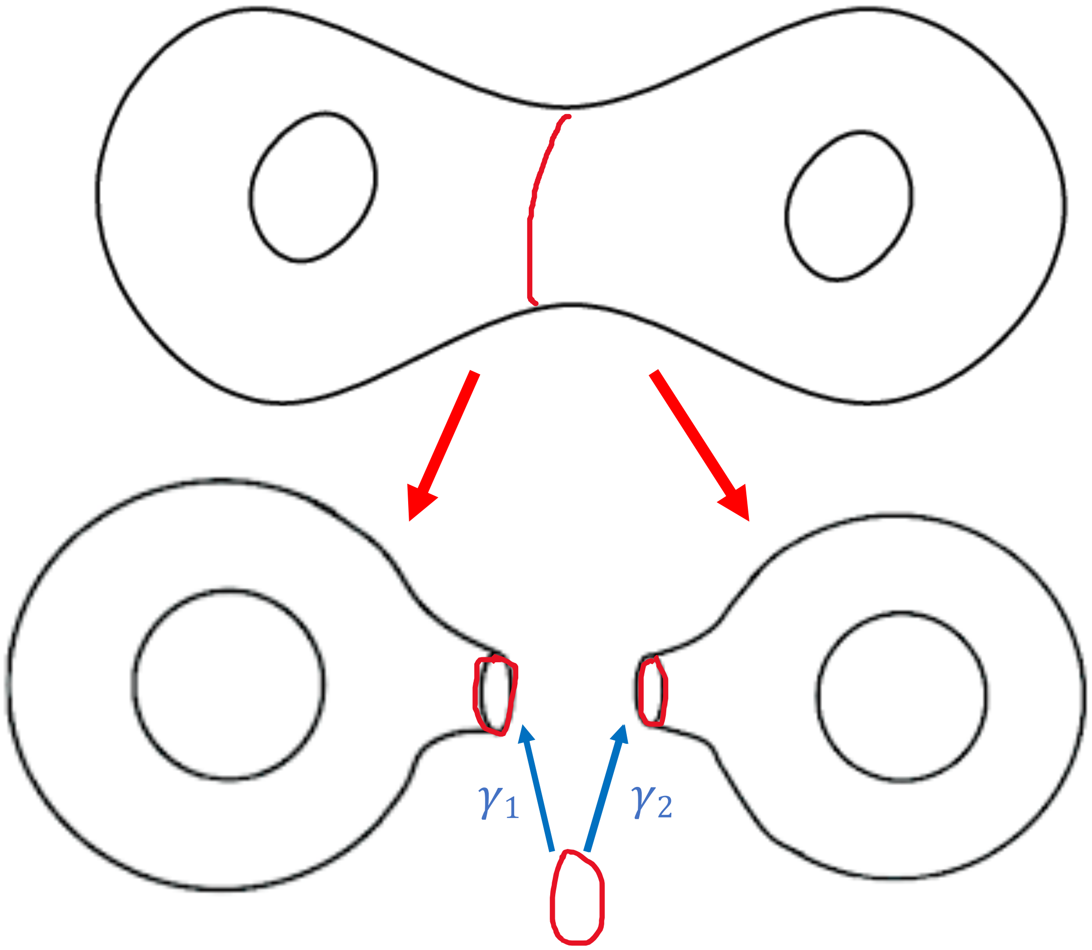

Theorem C (H. – Löh, 2020, [HLa]):
Let G be a finitely presented group with $H_2(G;\mathbb{R}) = 0$ and let $g \in [G,G]$ be an element in the commutator subgroup. Set

$$\tilde{G} = G_1 \star_{g1 = g2} G_2$$

where G_1, G_2 are copies of G and g1,g1 are copies of g. Then there is an *integral* class $\alpha \in H_2(\tilde{G};\mathbb{R})$ such that

$$||\alpha||_1 = 8 \cdot scl(g).$$

Proof of Theorem A:
2. Find groups with interesting stable commutator length

SCL relates to:

1. Surfaces
2. Quasimorphisms (Bavard)
3. Rotation Numbers (Ghys)

SCL is computable in

1. free groups (Calegari)
2. certain amalgamated free products (Chen)
3. certain groups which act on the circle

Proof of Theorem A:
2. Find groups with interesting stable commutator length

Theorem D (H. – Löh, 2020, [HLa], [HLb]):
There are finitely presented groups G_α with $H_2(G_\alpha; \mathbb{R}) = 0$ and an element $g_\alpha \in G_\alpha$ such that

$$scl(g_\alpha) = \alpha,$$

where

1. $\alpha \in \mathbb{Q}^+$

2. $\alpha = \frac{\arccos(1-2^n)}{2 \cdot \pi}, n \in \mathbb{N}^-$

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1. $\alpha \in \mathbb{Q}^+$ Universal central extension of Thompsons group T

2. $\alpha = \frac{\arccos(1-2^n)}{2 \cdot \pi}, n \in \mathbb{N}^-$ Certain central extension of $SL_2(\mathbb{Z}\left[\frac{1}{2}\right])$

Proof of Theorem A:
3. From 2-classes to 4-classes using products

Theorem E (H. – Löh, 2020, [HLa]):
For any 2-class $\alpha \in H_2(G;\mathbb{R})$ we have that the 4-class $\alpha \times \Sigma$ in $G \times \pi_1(\Sigma)$ has l^1 norm

$$||\alpha \times \Sigma||_1 = \frac{3}{2}||\alpha||_1 \cdot ||\Sigma||_1$$

for any surface Σ.

Proof:

$\leq$ immediate from [BK08],
$\geq$ via bounded cohomology, construct extremal cocycle analogous to [BK08].

Proof of Theorem A:
4. From 4-classes in groups to manifolds: Thom Realisation

Theorem (Thom):
For any integral 4-class $\alpha \in H_4(G; \mathbb{R})$ in a finitely presented group *G* there is a 4-manifold M such that

$$||M|| = ||\alpha||_1$$

Proof of Theorem A:
Summary

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□

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Input:
Tolerance $\epsilon > 0$

Out:
Interval $[q, q + \epsilon]$
with $\alpha \in [q,q+\epsilon]$

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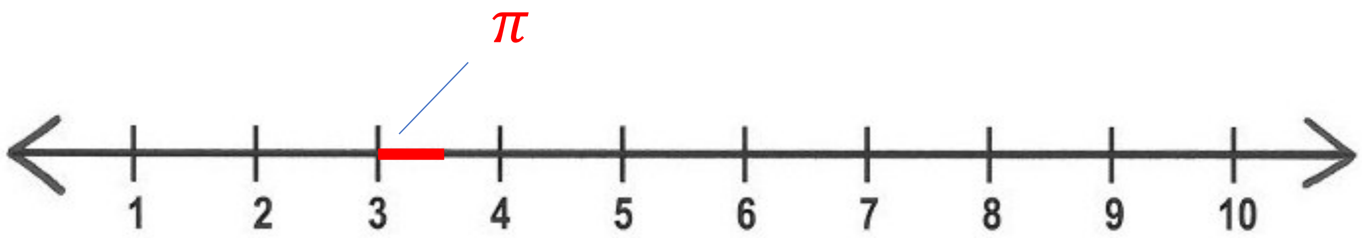
$\epsilon = \frac{1}{2}$

$Example: \alpha = \pi$

[3,3.5]



Right Computable numbers

$\alpha \in \mathbb{R}$ such that

Input:
Integer $n \in \mathbb{N}$

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Number $\alpha_n \geq 0$
s.t.
$(\alpha_n)_n$ descending
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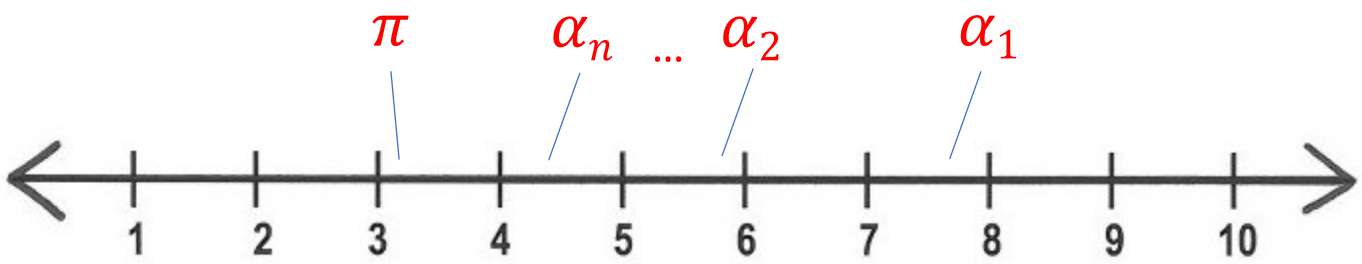

What is $SV(4)$?

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$\supseteq$

SCL

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RC^+ SCL

H., [H] = $\supseteq$ immediate

SCL^{rf}

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- RC^+ is the set of right computable numbers

What is $SV(4)$?

$$SV(4)$$

H. Löh, [HLb]

$$RC^{+} \subseteq SV(4) \supseteq SCL$$

$$RC^{+} = SCL^{rf} \subseteq SCL$$

$$SCL^{rf}$$

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References

- [BK08]: Michelle Bucher-Karlsson. The simplicial volume of closed manifolds covered by H × H . J. Topol., 1(3):584–602, 2008.
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