RAAGs and SCL

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Algebra, Geometry and Topology seminar
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Stable Commutator Length

	elements	chains
Objects	$\gamma: S^1 \to X$ $\gamma \in [\pi_1(X), \pi_1(X)]$	$\gamma_{\mathrm{i}}: S^1 \to X$ for $1 \le i \le m$ $\gamma_1 \cdots \gamma_m \in [\pi_1(X), \pi_1(X)]$
scl		
Example		

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scl	$\Phi: \Sigma \to X$, were Φ on $\partial\Sigma$ restricts to γ with degree $\mathrm{n}(\Phi)$ $scl(\gamma) := \inf \frac{-\chi(\Sigma)}{2\, n(\Phi)}$	$\Phi: \Sigma \to X$, were Φ on $\partial\Sigma$ restricts to γ with degree $\mathrm{n}(\Phi)$ $scl(\gamma_1 + \cdots + \gamma_n) := \inf \frac{-\chi(\Sigma)}{2\, n(\Phi)}$

Example

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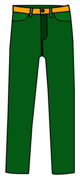
Example	$X = \Sigma_{1,1} =$  $\gamma = \partial\Sigma_{1,1}$ $\Phi = id : \Sigma_{1,1} \to \mathrm{X}$ $scl(\gamma) := \inf \frac{-\chi(\Sigma)}{2\, n(\Phi)} \leq -\frac{-1}{2} = \frac{1}{2}$	$X = \Sigma =$  $\gamma = \partial\Sigma$ $\Phi = id : \Sigma \to \mathrm{X}$ $scl(\gamma) \leq -\frac{-1}{2} = \frac{1}{2}$

Basic Properties

- Monotone: If $\Phi: G \to H$ is homomorphism then $scl_G(g) \geq scl_H(\Phi(g))$. Same for chains.
- Preserved under automorphisms.
- Preserved under conjugation.
- Invariant under retractions.
- Relationship between chains and elements: if $\mathrm{g}, \mathrm{h} \in G$:

$$scl(g+h) = scl(g\ t\ h\ t^{-1}) + \frac{1}{2}$$

in $G \star \langle t \rangle$.

- 'Linear Norm': $scl(g^n) = n \cdot scl(g)$, $scl(c_1 + c_2) \leq scl(c_1) + scl(c_2)$.

RAAGs

Γ : a graph with vertices V and edge set E.

A(Γ) = $\langle v \in V \mid [v, w],$ *for every* $(v, w) \in E \rangle$

RAAGs
and Graph Groups

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G(Γ) = $\langle g_v \in G_v \mid [g_v, g_w],\ \textit{for every}\ (v,w) \in E\ \rangle$

A(. . . .) = ?

A(⊠) = ?

A(□) = ?

A(⬡) = ?

A(⌂) = ?

RAAGs
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A(. . . .) = F_4

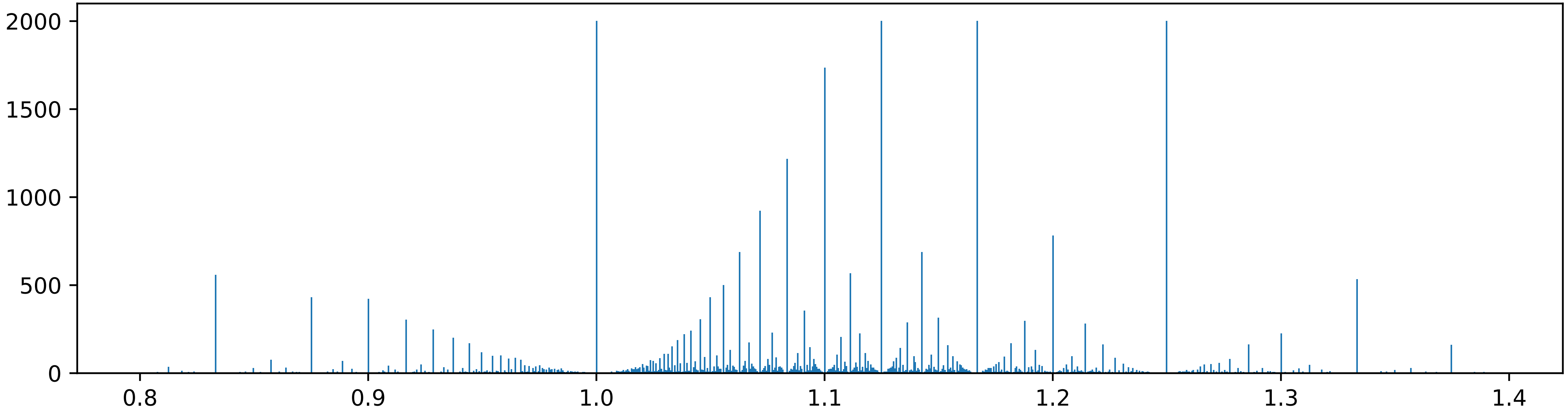
A(☒) = Z_4

A(☐) = $F_2 \times F_2$

A(⬠) = ?

A(⌂) = ?

SCL on Free Groups



Histogram of scl for 50'000 random words in F_2 of length 24.

	Free Group	RAAGs
Gaps g: element c: chain	$scl(g) \geq \frac{1}{2}$ (Duncan-Howie '91) $scl(c) \geq \frac{1}{8}$ (Tao '16) sharp?	$scl(g) \geq \frac{1}{2}$ (H. '18) $g \in A(\Gamma)$?
Spectrum	• Second gap? • Every rational ≥ 1?	?
Distribution	?	?
Complexity	scl: Computable in polynomial time (Calegari) cl: is NP complete. (H. '20)	? cl: NP Hard

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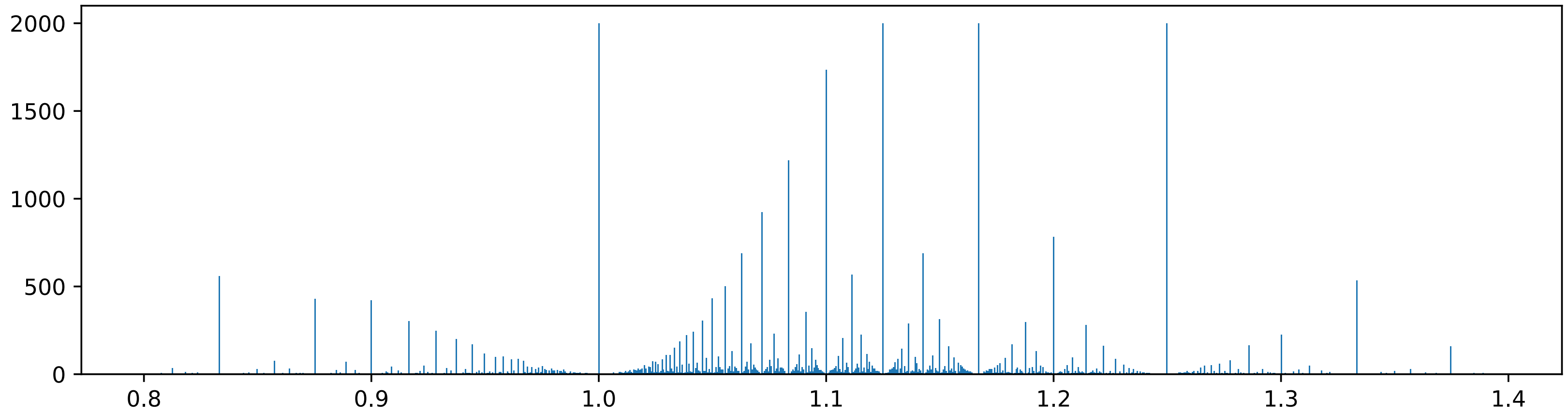
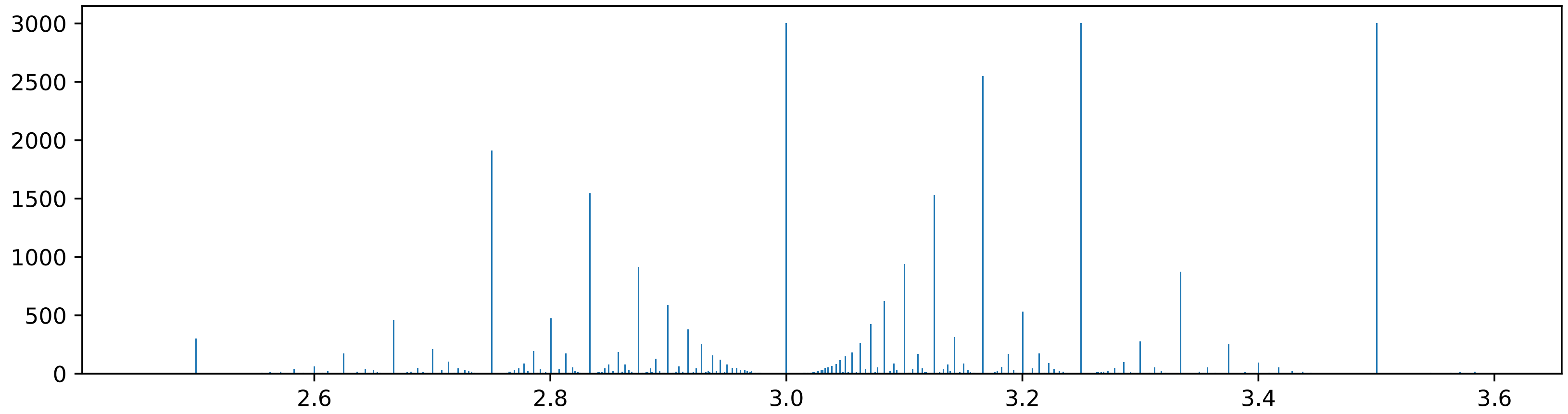
Gaps g: element c: chain	$scl(g) \geq \frac{1}{2}$ (Duncan-Howie '91) $scl(c) \geq \frac{1}{8}$ (Tao '16) sharp?	$scl(g) \geq \frac{1}{2}$ (H. '18) $g \in A(\Gamma)$ $scl(c) \geq \frac{1}{24\,\Delta(\Gamma) + 12}$ And $\exists d$, chain such that $scl(d) \leq \frac{1}{\Delta(\Gamma)}$.
Spectrum	• Second gap? • Every rational ≥ 1?	Every rational ≥ 1 is scl of some RAAG chain. (with Quasimorphisms!)
Distribution	?	Related to 'Fractional Stability Number'
Complexity	scl: Computable in polynomial time (Calegari) cl: is NP complete. (H. '20)	scl: NP Hard cl: NP Hard

(Chen – H. '20)

Upshot: SCL vs FSN



scl of random elements on F_2



scl of a fixed element of RAAGs on random graphs.

SCL Gaps

Known results:

- Free groups (Duncan-Howie, Tao)
- Amalgamated free products and graph of groups. (Chen – H.)
- Hyperbolic Groups (Fujiwara – Calegari)
- Mapping Class Groups (Bestvina – Bromberg – Fujiwara)
- BS groups and certain amalgamated free products (Clay– Louwsma – Forester)

Chains: Warmup

$$G = F_2 \times F_2 = \langle a, b\rangle \times \langle c, d\rangle$$

- $scl(a + b + AB) = scl(b + a + AB)$

Chains c, c' can have the same scl for 'trivial' reasons:

- Reordering terms,

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- $scl(a^2 + A - A) = scl(a + a + A + A) = 0$

Chains c, c' can have the same scl for 'trivial' reasons:

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- Conjugating terms,
- Replacing a term g^m by $m \cdot g$, and

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- $scl(a + a\,b\,A + AB) = scl(a + b + AB)$
- $scl(a^2 + A\ - A) = scl(a + a + A + A) = 0$
- $scl(a + c + AC) = scl(a + c + A + C) = 0$

Chains c, c' can have the same scl for ‘trivial’ reasons:

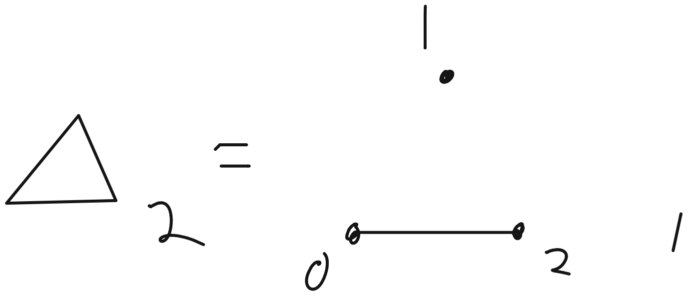
- Reordering terms,
- Adding $g + g^{-1}$,
- Conjugating terms,
- Replacing a term g^m by $m \cdot g$, and
- Replacing $g \cdot h$ by $g + h$, if g and h commute.

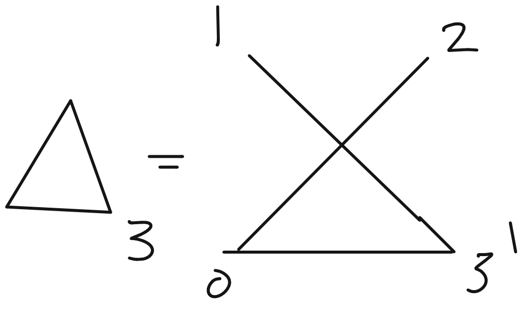
*Definition: c and c’ are **equivalent** if the same up to the above manipulations.*

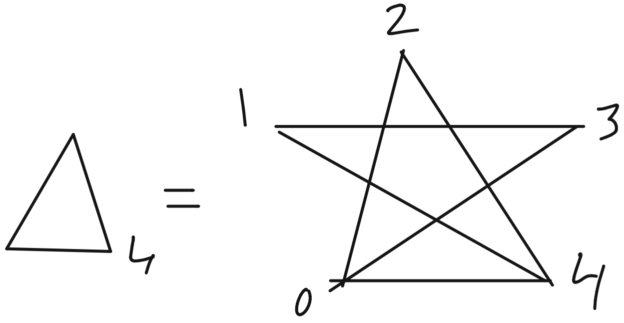
Graph Products: Warmup

Definition: A chain in a graph product is a ***vertex chain*** *if all its terms are supported on the vertex groups.*

Definition: The ***opposite path of length m, $\boldsymbol{\Delta_m}$*** *is the graph on vertices* $\{0, \dots, m\}$ *with edges whenever* $|i - j| \geq 2$.







Graph Products: Warmup

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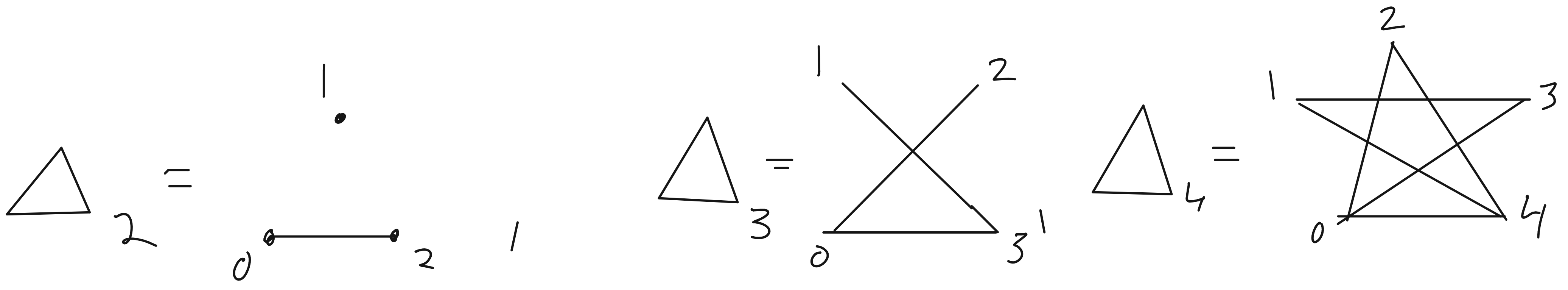
*Definition: The **opposite path of length m, $\boldsymbol{\Delta_m}$** is the graph on vertices $\{0, \dots, m\}$ with edges whenever $|i-j| \geq 2$.*

For a graph Γ,

$$\Delta(\Gamma) = \mathrm{m}ax\{m \mid \Delta_m\ is\ a\ full\ subgraph\ of\ \Gamma\},$$

*denotes the **opposite path length of $\boldsymbol{\Gamma}$**. In particular, $\Delta(\Delta_m) = m$.*



Main Result

Theorem (Chen – H. '20)
Let Γ *be a graph and G(*Γ*) be a graph product and let c be a chain on G(*Γ*).*

- *If* $scl(c) \leq \frac{1}{12\,\Delta(\Gamma)+24}$*, then c is equivalent to a vertex chain.*

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- *If $scl(c) \leq \frac{1}{12\,\Delta(\Gamma)+24}$, then c is equivalent to a vertex chain.*
- *There is a chain d in G(Γ) which is not equivalent to a vertex chain with $scl(d) \leq \frac{1}{\Delta(\Gamma)}$.*

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- *If* $scl(c) \leq \frac{1}{12\,\Delta(\Gamma)+24}$, *then c is equivalent to a vertex chain.*
- *There is a chain* d *in G(Γ) which is not equivalent to a vertex chain with* $scl(d) \leq \frac{1}{\Delta(\Gamma)}$.
- *There is an algorithm to compute* scl *on vertex chains.*

Overview of Proof of the Gaps Result

1. Gap for amalgamated free products $\mathrm{G} = A \star_C B$.
   - No long overlaps:

     Let $c = g + h$ be a chain such that there is no N such that g^N subword of h^∞: Gap of $\frac{1}{12N}$.

   - Def: $H < G$ is :
     - Malnormal: if for all $\forall g \in G \setminus H, h \in H$:, $g\, h\, g^{-1} \notin H$.
     - Central: $\forall g \in G \setminus H, H \in H$: $g\, h\, g^{-1} = h$.
     - CM subgroup: $\forall g \in G, \exists\ g' \in H\, g\, H$: for all $h \in H$: either $g' h\, g'^{-1} = h$, or $g' h\, g'^{-1} \notin H$.

     'Theorem': If $C < G$ is a CM subgroup + centralizer (of centralizer)^N is CM subgroup. Then G has no long overlaps of length N.

2. Gaps for Graph Groups. Write $G(\Gamma) = G\big(st(v)\big) \star_{Lk(v)} G(\Gamma \setminus v)$.

Chains with small scl

Let $d_0, \ldots, d_m$ be the generators of Δ_m.

Define $g_{i,j} = d_i \cdots d_j$. Then

Claim:
$g_{0,m}^m = g_{0,m-1}^m c$, *and*
$g_{1,m}^m = g_{1,m-1}^m c$.

Thus: d = $g_{0,m} - g_{0,m-1} - g_{1,m} + g_{1,m-1}$

$scl(m \cdot d) = scl\left(g_{0,m}^m - g_{0,m-1}^m - g_{1,m}^m + g_{1,m-1}^m \right)$
$scl(m \cdot d) = scl(g_{0,m}^m - g_{0,m-1}^m + c - g_{1,m}^m + g_{1,m-1}^m - c)$
$scl(m \cdot d) \leq scl(g_{0,m}^m - g_{0,m-1}^m + c) + scl(g_{1,m}^m - g_{1,m-1}^m + c)$
$scl(m \cdot d) \leq 1$

$$scl(d) \leq \frac{1}{m}$$

SCL on vertex chains

Question: Let $G(\Gamma)$ *be a graph product and let c be a chain* $c = \sum_v g_v$ *where* $G_v = F(a_v, b_v)$ *and* $g_v = [a_v, b_v]^2, i.e.\ scl(g_v) = 1.$ *What is* $scl(c)$*? Call it* $s(\Gamma)$*.*

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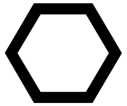
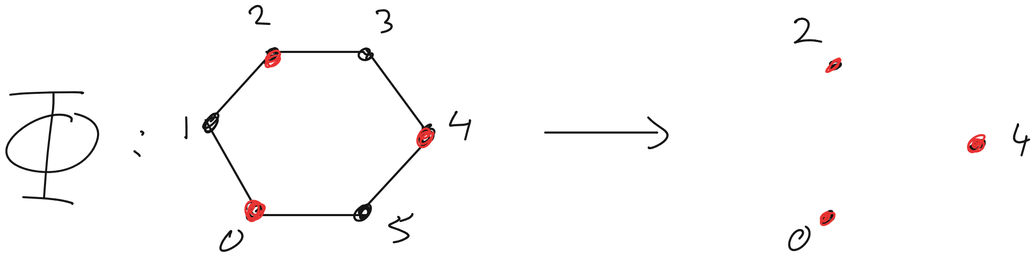
Examples:

Γ	$s(\Gamma)$
Complete graph?	1
Graph on n vertices without edges?	n
	?
	?

Special Case: 

Lower bound:



Special Case: ⬡

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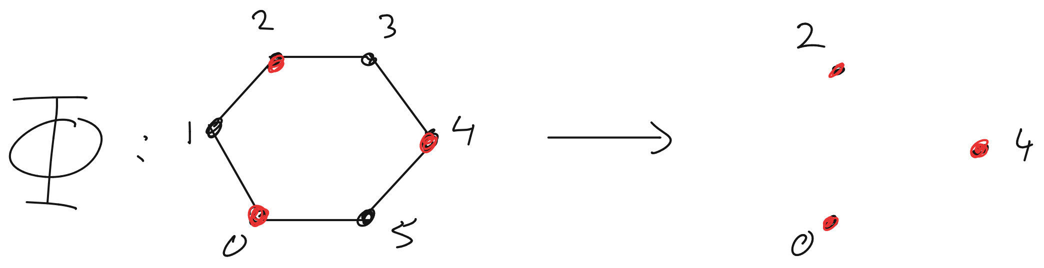


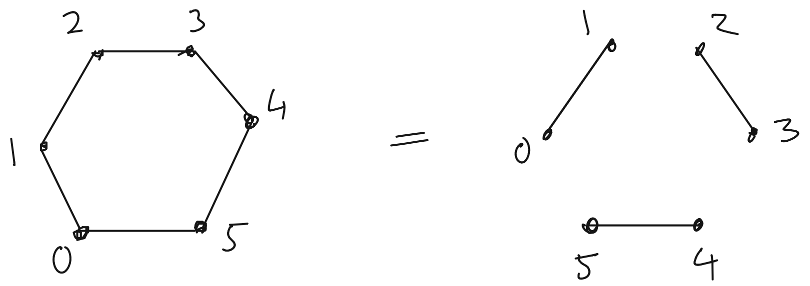
Generally: $S \subset \Gamma$ is called **stable,** if no vertices in S are connected. Call $\mathrm{sn}(\Gamma)$ the largest size of a maximal set (also: independence number, stability number).

$$s(\Gamma) \geq sn(\Gamma).$$

$$s(⬡) \geq 3$$

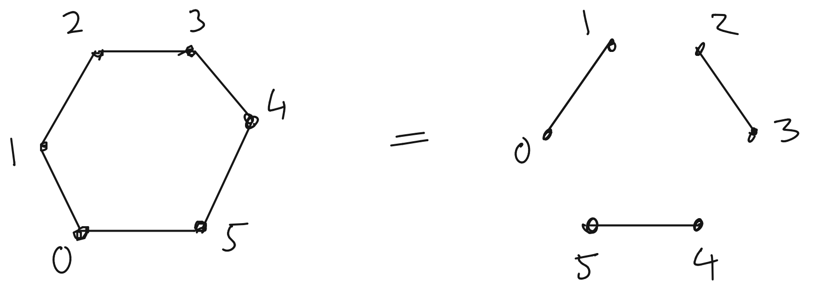
Special Case: ⬡

Upper bound:



Special Case: ⬡

Upper bound:



Generally: A **clique cover of $\boldsymbol{\Gamma}$** is a decomposition of Γ into cliques. A clique cover number ccn(Γ) is the smallest number of cliques need to cover Γ.

$$ccn(\Gamma) \geq s(\Gamma).$$

$$3 \geq s(\hexagon)$$

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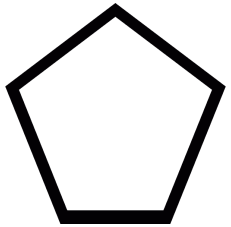
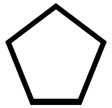
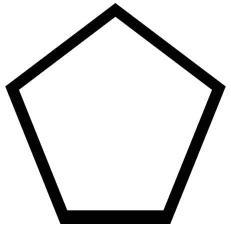
Special Case: 

$$ccn(\Gamma) \geq s(\Gamma) \geq sn(\Gamma).$$

$3 \geq s($  $) \geq 2$

Special Case: 

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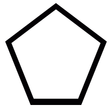
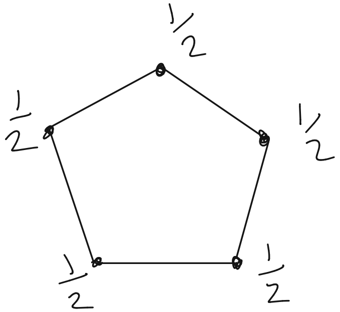
$3 \geq s($  $) \geq 2$

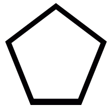
*Definition: A **fractional stable set of** $\boldsymbol{\Gamma}$ is a collection $\{s_v\}$ of non-negative real numbers for every $v \in V$**,** such that for every clique $C \subset \Gamma$**:** $\sum_{v \in C} s_v \leq 1$.*

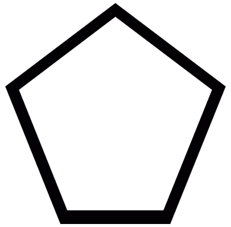
$$\boldsymbol{fsn(\Gamma) = max \sum s_v},$$

where maximum is taken over every fractional stable set.

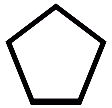


$$s(\Gamma) \geq fsn(\Gamma).$$

$s($  $) \geq 2.5$

Special Case: 

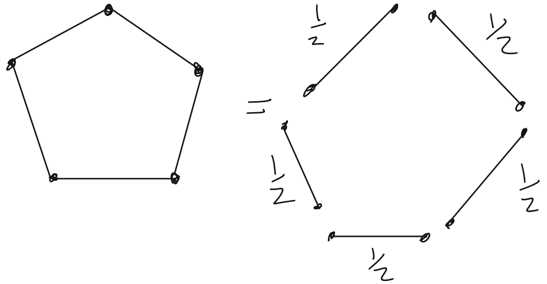
$$ccn(\Gamma) \geq s(\Gamma) \geq sn(\Gamma).$$

$3 \geq s($  $) \geq 2$

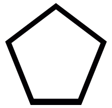
Definition: A ***fractional clique cover of*** $\mathbf{\Gamma}$ *is a collection* $\{s_C\}$ *of non-negative real numbers for every clique* c *of* Γ*, such that for every vertex* $v \in V$: $\sum_{v \in C} s_C \geq 1$.

$$\boldsymbol{fcc(\Gamma) = min \sum s_C,}$$

where minimum is taken over all fractional clique numbers.



$$fcc(\Gamma) \geq s(\Gamma).$$

$2.5 \geq s($  $)$

SCL on vertex chains

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Examples:

Γ	$s(\Gamma)$
Complete graph?	1
Graph on n vertices without edges?	n
	3
	2.5

Fractional Stability Number

Theorem (Chen – H. '20): For every graph Γ, $fsn(\Gamma) = s(\Gamma) = fcc(\Gamma)$.

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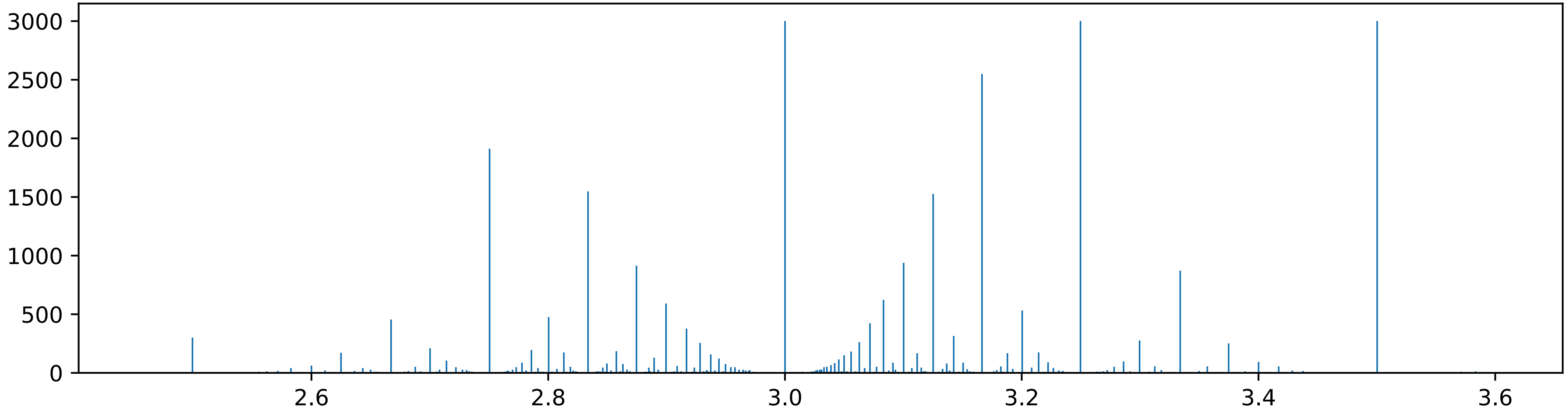
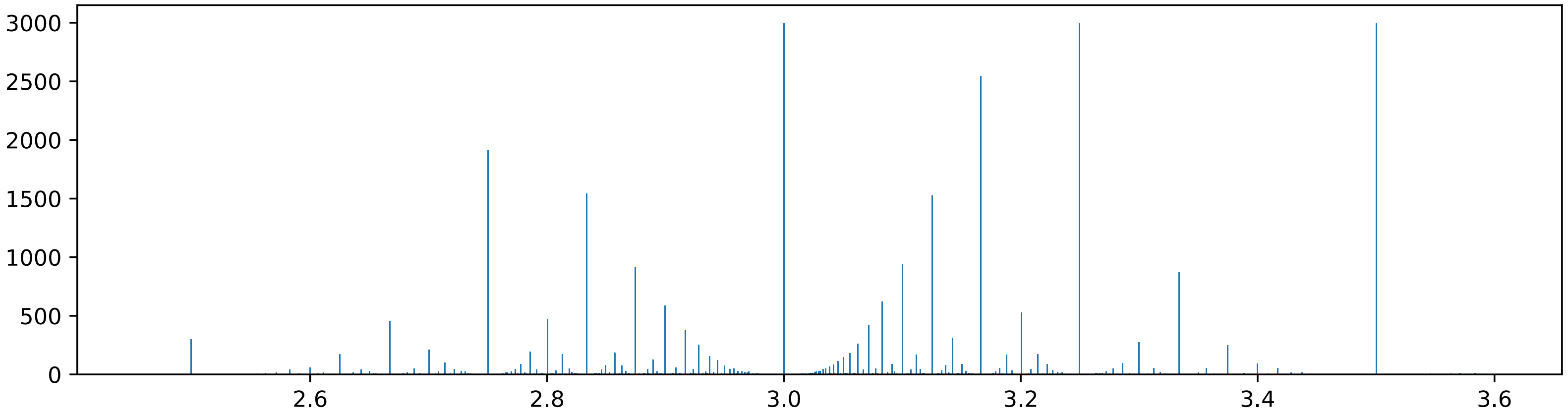


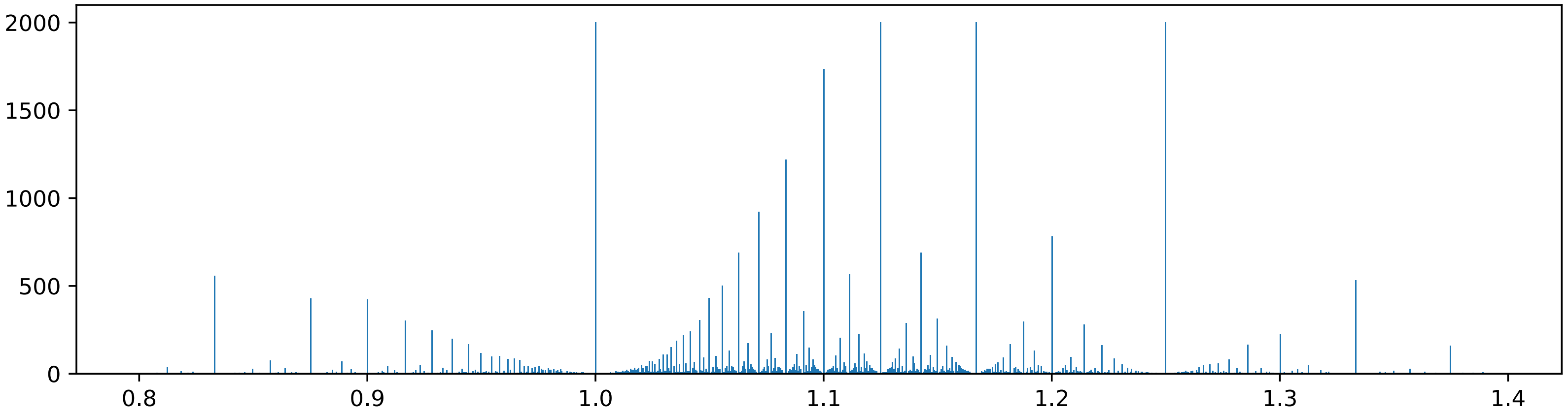
We have:

- $fsn\big(K(n,m)\big) = n/m$, where $K(n,m)$ is the opposite Kneser graph, thus every rational ≥ 1.
- fsn is NP hard (Subhash Khot)
- Relationship to (the better studied) Fractional Chromatic Number.

Modelling SCL and FSN

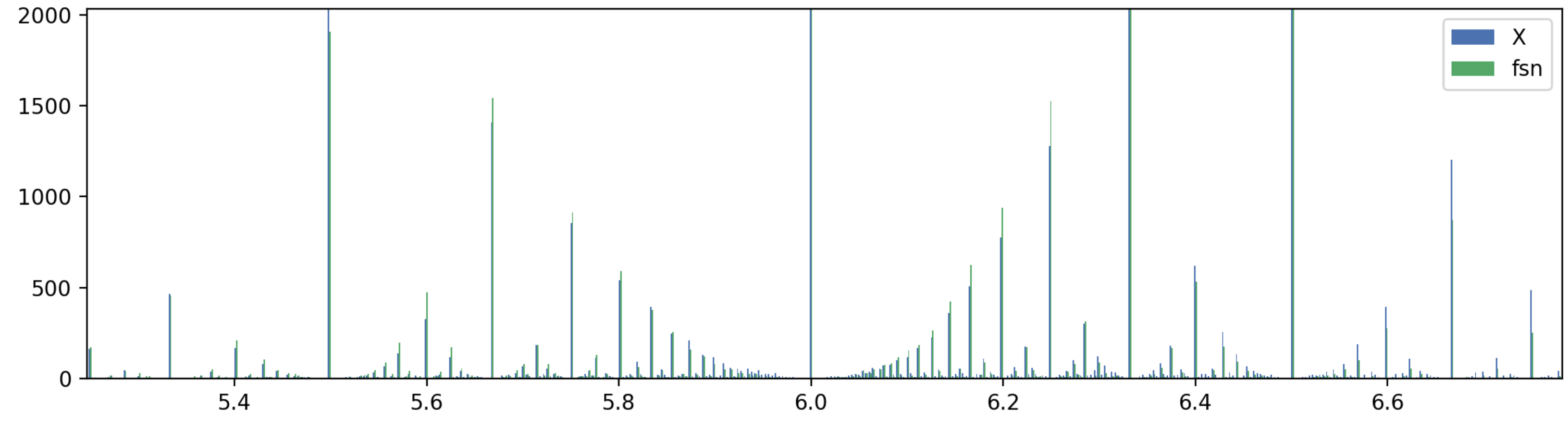


FSN of random graphs on 25 vertices

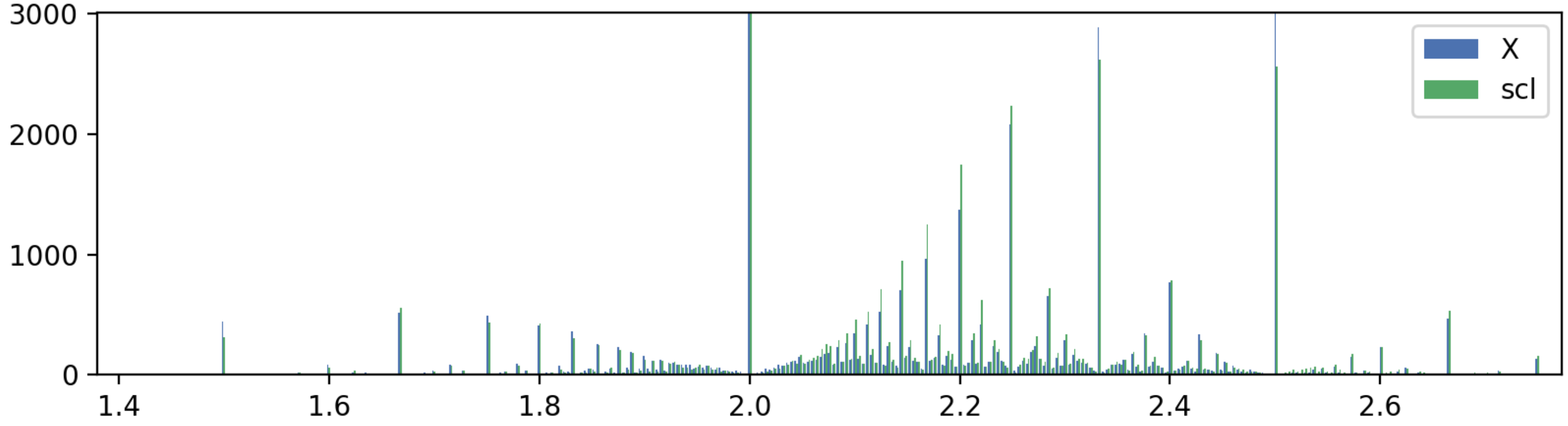


SCL of random elements of length 24 in F_2

Modelling SCL and FSN



FSN of random graphs on 25 vertices vs. X



SCL of random elements of length 24 in F_2 vs. X

X vs SCL and FSN

1. Choose with some probability an integer n
2. Choose a binomal random variable around a fixed mean μ

X vs SCL and FSN

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2. Choose a binomal random variable around a fixed mean μ

Choose $X(d, \beta, \mu, c_1, c_2)$ as follows:

1. Choose an integer n with probability proportional to $n^{(1-\, n^{\beta})\cdot d}$
2. Let N_n be the random variable with distribution $N(\mu, c_1 \cdot n^{c_2})$ and round to nearest element in $\frac{1}{n}$Z

Question/Conjecture: Is there a natural distribution which models both SCL and FSN?

Open Questions

1. *Is scl computable (and rational) on RAAGs?*
2. *What is the distribution of scl?*
3. *Is there a scl gap for special groups?*

End

Quasimorphisms

Definition: A map $\phi: G \to R$ *is a **homomorphism** if* *for all* $g, h \in G$:

$$|\phi(g) + \phi(h) - \phi(gh)| = 0$$

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Theorem (Bavard, Calegari): For every $g \in [G, G]$ $(c, chain)$*:*

$$scl(g) = \sup_{\phi} \frac{\phi(g)}{2\,D(\phi)}$$

where the supremum is taken over all homogeneous quasimorphisms.

FSN and quasimorphisms

- Given: $c = \sum_v c_v$, such that $scl(c_v) = 1$.
- Let $\phi_v: G_v \to R$ be a collection of extremal quasimorphisms to c_v for every $v \in V$.
- $\{s_v\}_v$ be the maximal fractional stable set. Then:

$$\phi(g) = \sum s_v \cdot \phi_v(g)$$

is an extremal quasimorphism for c, for g cyclically reduced.

End

Stable Commutator Length

	Algebraic	Geometric
Objects	$g \in [G, G]$	
Invariants	$cl(g) := \min\{n \mid g = [x_1, y_1] \cdots [x_n, y_n]\}$ $scl(g) \coloneqq \lim_{\{n \to \infty\}} cl(g^n)/n$	
Example		

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Example	$G = F_2, g = [a,b]$ $cl([a,b]) = 1$ $cl([a,b]^3) = 2$ $cl([a,b]^n) = \lceil \frac{n+1}{2} \rceil$ $scl([a,b]) = \frac{1}{2}$	

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Objects	$g \in [G, G]$	$\gamma: S^1 \to X$ $\gamma \in [\pi_1(X), \pi_1(X)]$
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Stable Commutator Length for chains

	Algebraic	Geometric
Objects	$c = g_1 + \cdots + g_n$ s.t. $g_1 \cdots g_m \in [G, G]$	
Invariants	$cl(g_1 + \cdots + g_m) = \min\{cl(t_1 g_1 t_1^{-1} \cdots t_m g_m t_m^{-1})$ $scl(g_1 + \cdots + g_m) \coloneqq \lim_{\{n \to \infty\}} cl(g_1^n + \cdots + g_m^n)/n$	
Example		

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Example	$G = F_2, g_1 = a, g_2 = b, g_3 = A\,B$ $c = g_1 + g_2 + g_3$ $cl(g_1 + g_2 + g_3) = 0$ $cl(g_1^3 + g_2^3 + g_3^3) = 1$ $cl(g_1^n + g_2^n + g_3^n) = \lceil \frac{n-1}{2} \rceil$ $scl(g_1 + g_2 + g_3) = \frac{1}{2}$	

Stable Commutator Length for chains

	Algebraic	Geometric
Objects	$c = g_1 + \cdots + g_n$ s.t. $g_1 \cdots g_m \in [G, G]$	$\gamma_{\mathrm{i}}: S^1 \to X$ for $1 \leq i \leq m$ $\gamma_1 \cdots \gamma_m \in [\pi_1(X), \pi_1(X)]$
Invariants	$cl(g_1 + \cdots + g_m) = \min\{cl(t_1 g_1 t_1^{-1} \cdots t_m g_m t_m^{-1})$ $scl(g_1 + \cdots + g_m) \coloneqq \lim_{\{n\to\infty\}} cl(g_1^n + \cdots + g_m^n)/n$	$\Phi: \Sigma \to X$, were Φ on $\partial\Sigma$ restricts to γ with degree $\mathrm{n}(\Phi)$ $scl(\gamma) := \inf \frac{-\chi(\Sigma)}{2\, n(\Phi)}$
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Stable Commutator Length for chains

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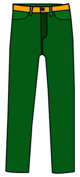
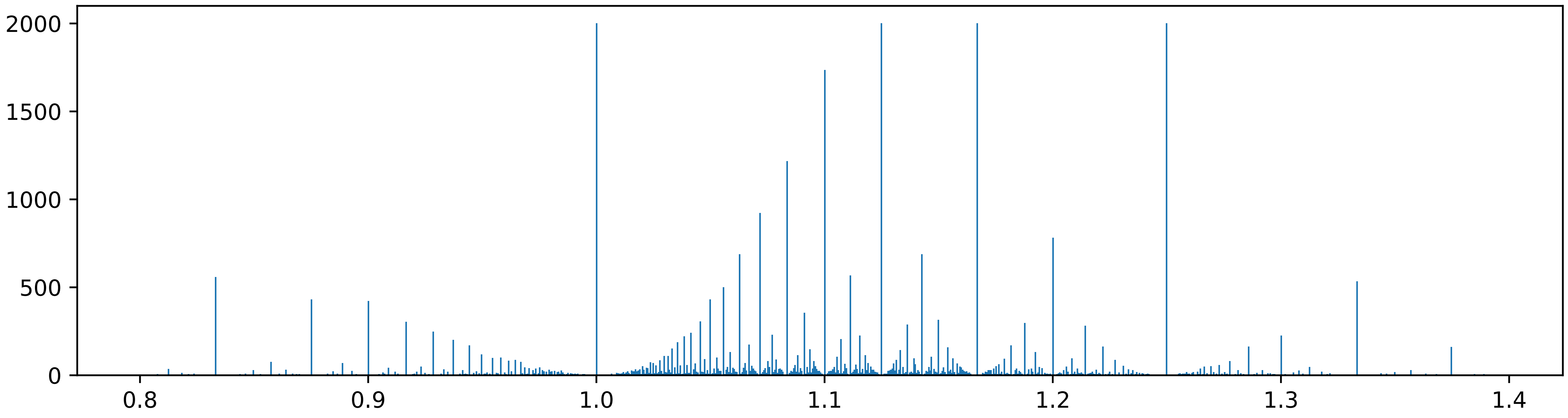
Objects	$c = g_1 + \cdots + g_n$ s.t. $g_1 \cdots g_m \in [G,G]$	$\gamma_i: S^1 \to X$ for $1 \le i \le m$, $\gamma_1 \cdots \gamma_m \in [\pi_1(X), \pi_1(X)]$
Invariants	$cl(g_1 + \cdots + g_m) = \min\{cl(t_1 g_1 t_1^{-1} \cdots t_m g_m t_m^{-1})$ $scl(g_1 + \cdots + g_m) := \lim_{\{n\to\infty\}} cl(g_1^n + \cdots + g_m^n)/n$	$\Phi: \Sigma \to X$, were Φ on $\partial\Sigma$ restricts to γ with degree $\mathrm{n}(\Phi)$ $scl(\gamma) := \inf \frac{-\chi(\Sigma)}{2\, n(\Phi)}$
Example	$G = F_2, g_1 = a, g_2 = b, g_3 = A\,B$ $c = g_1 + g_2 + g_3$ $cl(g_1 + g_2 + g_3) = 0$ $cl(g_1^3 + g_2^3 + g_3^3) = 1$ $cl(g_1^n + g_2^n + g_3^n) = \lceil \frac{n-1}{2} \rceil$ $scl(g_1 + g_2 + g_3) = \frac{1}{2}$	$X = \Sigma =$  $\gamma = \partial\Sigma$ $\Phi = id : \Sigma \to \mathrm{X}$ $scl(\gamma) \le -\frac{-1}{2} = \frac{1}{2}$

SCL on Free Groups



What's known:

- There is a fast (polynomial time) algorithm to compute scl on single elements and chains (Calegari)
- SCL is rational (Calegari)
- There is a gap of 1/2 for single elements and 1/8 (sharp?) for chains. (Duncan—Howie and Tao)

Open Questions:

- What's the exact gap for chains?
- Is there a second gap for single elements?
- Are all rationals greater than 1 realized as scl?
- Explain the distribution.
- Quasimorphisms?